fluxions, especially of orders higher than the first, as well as against matter." To this De Morgan replied: "I have no doubt Berkeley knew that fluxions were sound enough." Berkeley himself said: "I have no controversy about your conclusions, but only about your logic and method." In view of the further fact that Berkeley in the *Analyst* advanced the theory of "Compensation of Errors" we incline to the opinion of De Morgan. The theory of "Compensation of Errors," we may add, was advanced independently by Lagrange and L. N. M. Carnot. According to Philip E. B. Jourdain this theory is found also in Maclaurin's *Fluxions*.

There are four other points in Berkeley's *Analyst* to which we desire to direct attention. First his protestation against dropping quantities because they are comparatively very small. Jurin in his first reply argues in favor of the rejection of infinitesimals. In his second reply, after having received a castigation from Berkeley, Jurin says that this part of his argument was intended for popular consumption, for men such as one meets in London, who, when told that if Sir Isaac Newton were to measure the height of St. Paul's Church by fluxions he would be out not more than one tenth of a hair's breatdh, and when further told that two books had been written in this controversy, would fly into a passion, would make reflections about "somebody's being overpaid," and would use expletives not fit for print.

Secondly, Berkeley's denial of the existence of infinitely small quantities is in conformity with the tenets of the recent school of Weierstrass and Georg Cantor.

Interesting is Berkeley's attack upon Newton's derivation of the *moment* or increment of a rectangle AB, as it is given in the *Principia*. Newton derives this moment by the difference $(A + \frac{1}{2}a)\ (B + \frac{1}{2}b) - (A - \frac{1}{2}a)\ (B - \frac{1}{2}b) = Ab + Ba$, where a and b are assumed to be the increments of the sides. Berkeley argues with conviction that the increment of the rectangle AB is $bA + aB + ab$. Jurin takes the arithmetical mean of the increment $bA + aB + ab$ of the rectangle AB and of the decrement $bA + aB - ab$ of AB and obtains the desired true increment or "moment" as $aB + bA$. Sir William Rowan Hamilton sided with Berkeley against Newton on this point but no eighteenth century mathematician in England admitted the validity of Berkeley's criticism.

Lastly we come to the most fundamental of Berkeley's criticisms of Newton which centers upon what is called Berkeley's *lemma*: If in a demonstration an assumption is made, by virtue of which certain conclusions follow, and if afterward that assumption is destroyed or rejected, then all the conclusions that had been reached by the first assumption must also be destroyed or rejected. Berkeley applied this *lemma* to Newton's mode of deriving the fluxion of x^n as given in the *Quadrature of Curves* of 1704. Newton gives x a finite increment o, expands $(x + o)^n$ by the binomial formula, subtracts x^n and divides the remainder by o. He then lets o be zero and obtains the fluxion nx^{n-1}. Berkeley says that this reasoning is not fair or conclusive. "For when it is said, let the increment be nothing, the former supposition that the increment be something is

destroyed and yet the expression got by that former supposition is retained. By Berkeley's *lemma*, this is a false way of reasoning, "such as would not be allowed of in Divinity."

It is interesting to observe that no British mathematician of the eighteenth century acknowledged the soundness of Berkeley's *lemma* and its application.

Jurin, in his second reply to Berkeley, argues against the *lemma* thus: "You say that if one supposition be made, and be afterwards destroyed by a contrary supposition, then everything that followed from the first supposition is destroyed with it." Not so, when the supposition and its contradiction are made at different times. "Let us imagine yourself and me to be debating this matter in an open field, . . . a sudden violent rain falls . . . we are all wet to the skin . . . it clears up . . . you endeavor to persuade me I am not wet. The shower, you say, is vanished and gone and consequently your wetness must have vanished with it." The first recognition in England of the soundness of Berkeley's *lemma* came in 1803 from Robert Woodhouse, who, in his *Principles of Analytical Calculation*, says that the methods of treating the calculus "all are equally liable to the objection of Berkeley, concerning the *fallacia suppositionis*, or the *shifting of the hypothesis*." In finding the fluxion of x^n, the binomial expansion is effected "on the express supposition, that o is some quantity, if you take o equal to zero, the hypothesis is, as Berkeley says, shifted and there is a manifest sophism in the process."

After Berkeley terminated his debate with the mathematicians, two mathematicians started a quarrel among themselves. Thus arose the second controversy on fluxions, which is comparatively little known.

Benjamin Robins, a self-educated mathematician, felt that Jurin had not entered a satisfactory defence of Newton, so Robins himself in 1735, published a tract entitled *A Discourse Concerning the Nature and Certainty of Sir Isaac Newton's Method of Fluxions and of Prime and Ultimate Ratios*. Robins makes no reference to Berkeley or Jurin, or to their controversy. He lays the foundation of the calculus upon the concept of a *limit*. He speaks of a *limit* as a magnitude "to which a varying magnitude can approach within any degree of nearness whatever, though it can never be made absolutely equal to it." Here for the first time is the stand taken openly, clearly, explicitly, that a variable can never reach its limit. From the standpoint of debating, this stand is a decided gain, but it is a gain made at the expense of generality. He descends to a very special type of variation which is not the variation encountered in ordinary mechanics; it is an artificial variation which does not permit Achilles to catch the tortoise. But this narrow concept of a limit nevertheless answers very well the needs of ordinary geometry. Robins's tract is remarkable for clearness and soundness of exposition; it is a marked advance in that respect. The use of infinitely small quantities is rigidly excluded. The objections raised by Berkeley against Fluxions did not apply to Robins's exposition. A long account of Robins's *Discourse*, prepared by Robins himself, was published in a London monthly called *The Present State of the Republick of Letters*. In the next number of this monthly appeared an article

by Jurin, under the pseudonym "Philalethes Cantabrigiensis," in which he says that in his debate with Berkeley he adhered strictly to Newton's language, but that some other defenders of Newton (meaning Robins) were guilty of departing from it. Jurin argues that the words *fiunt ultimo æquales* used by Newton in Lemma I of Book I in the *Principia*, mean that the quantities (the inscribed and circumscribed polygons) "do at last become actually, perfectly, and absolutely equal"; in modern phraseology, the limit is reached. Several passages in the writings of Newton are examined and many illustrations are given. Robins prepared a reply to Jurin, and thus a controversy had gotten under way which threatened at one time to become endless. For two years there was a steady stream of articles in the *Republick of Letters* and its successor *The Works of the Learned*. Pemberton entered the controversy during the second year on the side of Robins, but contributed nothing of value. About twenty articles were written, one of which filled one hundred and thirty-six pages. All articles taken together covered over seven hundred printed pages. They were attempts to ascertain what Newton's ideas of *fluxions* and *moments* were, and whether Newton meant that a variable can reach its limit or cannot. And a good part of this material has escaped the attention of mathematical historians until now. The first few articles displayed care and ability, the later articles suffered in scientific value from the excessive heat of controversy. Jurin's articles against Robins are superior to his articles against Berkeley. The debate is the most thorough discussion of the theory of limits carried on in England during the eighteenth century. It constitutes a refinement of previous conceptions. In my judgment Jurin's interpretation of Newton was more nearly correct than that of Robins. The two disputants examined and reëxamined every passage of Newton's printed papers bearing on fluxions. Robins saw in Newton's condensed writings only variables which do not reach their limits; Jurin insisted that Newton permitted variables to reach their limits. Jurin admitted the calculus could be consistently founded upon Robins's idea of a limit, but he also insisted that Robins misrepresented Newton. Jurin's conceptions were quite broad for his time. He said: "Now whether a quantity or a ratio shall arrive at its limit or shall not arrive at it, depends entirely upon the supposition we make of the time during which the quantity or ratio is conceived constantly to tend or approach towards its limit." In other words, whether a variable reaches its limit or not is a matter of choice. We may impose conditions, so that the variable reaches its limit, or conditions under which it does not reach its limit. Thus Jurin was perhaps the first consciously to modify and generalize the limit concept. Modifications and generalizations of this have been going on ever since and are still in progress. A serious difficulty in permitting variables of the kind ordinarily arising in geometry to reach their limits lay in the fact that the imagination is not able to follow the variable through an infinity of steps that lead into the limit. The imagination exhausts itself in the effort. It is right here that Robins's variables which do not reach their limits had a great advantage. Jurin took great pains to devise illustrations of limit-reaching variables, intended to

aid the imagination, though, as he admits, incapable of exhibiting the process "all the way." In one place Jurin says: "Since Mr. Robins is pleased to talk so much about straining our imagination, . . . let us see if we cannot find some plain and easy way of representing to the imagination that actual equality, at which the inscribed and circumscribed figures will arrive with each other, and with the curvilineal figure, at the expiration of the finite time." His procedure amounts to expressing the inscribed and circumscribed polygons as functions of the time, such that the limit is reached in a finite time.

It is interesting that, toward the end of his long debate with Robins, Jurin begins to disavow infinitely small quantities. He brings out the difference between infinitesimals as variables, and infinitesimals as constants. He rejects all quantity "fixed, determinate, invariable, indivisible, less than any finite quantity whatsoever," but he usually admits somewhat hazily a quantity "variable, divisible, that, by a constant diminution, is conceived to become less than any finite quantity whatever, and at last to vanish into nothing."

Soon after the Berkeley onslaught, there appeared nine British texts on fluxions, only one of which was of decidedly inferior type. None of these texts refer to the Jurin-Robins dispute. The latter was not widely noticed. The two thought-compelling publications that were widely read were Newton's *Quadrature of Curves* of 1704, and Berkeley's *Analyst*. The latter tract was always criticized by the mathematicians, yet always held in awe. These two tracts, together with Robins's *Discourse*, and Maclaurin's celebrated work on *Fluxions*, which appeared in 1742, mark the highest point of logical precision reached in England during the eighteenth century. All three of the great sections of the British Isles had contributed to this end: England through Newton and Robins, Ireland through Berkeley, and Scotland through Maclaurin. Maclaurin was familiar with the writings of the other three. He took the Greek demonstrative rigor as a model. In a biography of Maclaurin it is stated that several years before the publication of his fluxions, his demonstrations had been communicated to Berkeley and "Mr. Maclaurin had treated him with the greatest personal respect and civility; notwithstanding which, in his pamphlet on tar-water, he (Berkeley) renews the charge, as if nothing had been done" to remove the logical difficulties. Maclaurin avoided the use of infinitely small quantities, "an infinitely little magnitude being," as he expressed himself, "too bold a postulatum for such a science as geometry." He laid less stress upon the concept of a limit than did Robins and Jurin, and followed more closely the kinematical concepts of Newton. The term velocity had been the subject of dispute between Berkeley and Walton. Maclaurin perceived the difficulty of arguing that variable velocity is a physical fact. He defined the velocity of a variable motion as the space that *would be described* if the motion had continued uniform. He also quotes Barrow: Velocity is the "power by which a certain space may be described in a certain time" and then explains "power" by the consideration of "cause" and "effect" in a way that sounds odd in a work on fluxions. However, when we think of the Thomson-Dirichlet Principle we must acknowledge that the eighteenth century was not

the only time when physical concepts were brought to the aid of mathematical theory. Apparently following Robins, Maclaurin's explanations imply that he does not encourage variables actually to reach their limits. Maclaurin secured his rigor of demonstration at a tremendous sacrifice. His work on *Fluxions* consists of seven hundred sixty-three pages; the first five hundred ninety pages do not contain the notation of fluxions at all; the mode of exposition is rhetorical. This part deals with the derivation of the fluxions of different geometric figures, of logarithms, of trigonometric functions, also with the discussion of maxima and minima, asymptotes, curvature and mechanics, in a manner that the ancients might have adopted and with a verbosity of which the ancients are guilty. The consequence was that the work was not attractive reading. It was much praised and much neglected. Fifty-nine years elapsed before a second edition appeared. As we shall see, the book did not stop disputes on fluxions.

The middle and latter part of the eighteenth century were periods of mediocrity. There appeared a dozen books on fluxions, of which those of Wllliam Emerson and Thomas Simpson were the most noted. Both Emerson and Simpson were self-educated mathematicians, possessing the strength and the weakness usual with such preparation. Emerson returned to the use of infinitely small quantities, but a fluxion was defined as a velocity. This return to the use of infinitely small quantities is noticeable in several English texts of the second half of the century. An old lady once defended Calvinism by saying that if you took away her total depravity you took away her religion. There were mathematicians who believed that if you took away infinitely small quantities you took away all their mathematics. Simpson, in his text of 1750, which is a thorough revision of his text of 1737, avoids the use of infinitely small quantities. His definition of fluxion is as follows: "The magnitude by which any flowing quantity *would be* uniformly increased in a given time, with the generating celerity at any proposed position, or instant (was it from thence to continue invariable), is the fluxion of the said quantity at that position or instant." Substantially this definition of a fluxion was adopted later by Charles Hutton. Simpson dodges the word velocity, and remarks: "If motion in (or at) a point be so difficult to conceive that *some* have gone even so far as to dispute the very existence of motion, how much more perplexing must it be to form a conception, not only of the velocity of a motion, but also infinite changes and affections of it, in one and the same point, where all the orders of fluxions have to be considered." Simpson's definition and treatment of fluxions avoided the fictitious infinitesimals, as well as the perplexing term "velocity." Nevertheless, it did not enjoy security against attack, but was fiercely criticized in the London *Monthly Review*. The critic claimed that it is objectionable to define fluxion as the "magnitude by which any flowing quantity would be uniformly increased," for it was argued, that "in quantities *uniformly* generated, the *fluxion* must be the *fluent* itself, or else a part of it." It was claimed that Simpson's endeavor to exclude velocity "cannot be made intelligible without introducing velocity into it." "Again he

mistakes the *effect* for the *cause;* for the thing generated must owe its existence to something, and this can only be the velocity of its motion, but it can never be the *cause itself,* as his definition would erroneously suggest." This obscure criticism of obscure points in Simpson's exposition initiated a third debate on fluxions which was carried on in the *Ladies Diary* and in ephemeral journals called the *Palladium,* the *Lady's Philosopher,* the *Mathematical Exercises* (edited by John Turner). The debate was carried on between friends of Emerson on one side and friends of Simpson on the other. Emerson and Simpson do not themselves appear in the controversy. The friends of Emerson published in 1752 in London an anonymous pamphlet, entitled *Truth Triumphant or Fluxions for the Ladies, Showing the Cause to be Before the Effect, etc.,* which was criticized by the friends of Simpson as a "scurrilous pamphlet." It contains much that is foolish, a few passages eulogizing the works of Emerson, but also critical considerations which are of some interest and disclose the need of a more satisfactory arithmetical continuum. All in all this debate was carried on upon a much lower scientific plane than the former debates. The debaters represented the rank and file of mathematicians.

In the second half of the century several abortive attempts at arithmetization of the calculus were made. The most worthy of these attempts is due to John Landen, but his analysis is so complicated as to be prohibitive. Towards the latter part of the eighteenth century the efforts at rigorous exposition, which were so conspicuous in the years 1735–1742, slackened more and more. Colin Maclaurin was seldom read and John Robins was altogether forgotten. William Hales's discovery of Robins's *Discourse* in 1804 astonished him as would the discovery of a new work of Archimedes. The first edition of the *Encyclopedia Britannica,* 1771, permitted a "fluxion" to degenerate into an "increment" acquired in "less than any assigned time." The same article on fluxions appeared in the second edition (1779) and in the third edition (1797). In 1801 there was published in London *Agnesi's Analytical Institutions,* which many years earlier had been translated by John Colson from the Italian into English. How Colson's conscience must have troubled him, when a fluxion stood out in his translation as something "infinitely small," may be judged by the consideration that in 1736 he brought out an English translation of Newton's *Method of Fluxions.* With Newton a fluxion always meant a finite velocity. We wonder what Robins and Maclaurin would have thought had they been alive in 1897 and 1901 and read these definitions. What horrible visions would these ghosts of departed quantities have brought to Bishop Berkeley had he been alive!

As we look back over the century we see that the eight years immediately following Berkeley's *Analyst* were *eight great years,* during which Jurin, and especially Robins and Maclaurin made wonderful strides in the banishment of infinitely small quantities and the development of the concept of a limit. Both before and after that period of eight years, there existed in most writings of the eighteenth century in Great Britain, a mixture of Continental and British conceptions of the new calculus, a superposition of British symbols and phraseology

upon the older Continental concepts. The result was a system, destitute of scientific interest. Newton's notation was poor and Leibniz's philosophy of the calculus was poor. That result represents the temporary survival of the least fit of both systems. The more recent international course of events has been in a diametrically opposite direction, namely, not to superpose Newtonian symbols and phraseology upon Leibnizian concepts, but, on the contrary, to superpose the Leibnizian notation and phraseology upon the limit-concept, as developed by Newton, Jurin, Robins, Maclaurin, D'Alembert and later writers.

About the opening of the nineteenth century more recent continental authors began to attract the attention of the English. Extensive accounts appeared in the London *Monthly Review* of Lagrange's *Theory of Functions*, Lacroix's *Differential Calculus*, Carnot's *Reflexions on the Metaphysics of the Infinitesimal Calculus*. These texts were compared with English publications in a way not altogether favoring the English. Finally in 1805 Robert Woodhouse of Caius College, Cambridge, brought out his *Principles of Analytical Calculation* which contained many keen criticisms of both Continental and British mathematicians. Woodhouse is the first English mathematician who had a good word for Berkeley. He said: "I cannot quit this part of my subject without commenting on the *Analyst* and the subsequent pieces, as forming the most satisfactory controversial discussion in pure science that ever yet appeared: into what perfection of perspicuity and logical precision the doctrine of fluxions may be advanced, is no subject of consideration; but view the doctrine as Berkeley found it, and its defects in metaphysics and logic are clearly made out. If for the purpose of habituating the mind to just reasoning . . . I were to recommend a book, it would be the *Analyst*."

Woodhouse is the forerunner in Cambridge of Babbage, Peacock, and the younger Herschel, in the promotion of the principles of pure *D*-ism in opposition to the dot-age of the university.

As usually happens in reformations so here there was discarded and lost not only what was antiquated, but also what was meritorious. Robins's *Discourse* of 1735, with its full and complete disavowal of infinitesimals and clear-cut, though narrow, conception of a limit was quite forgotten and D'Alembert's definition was recommended and widely used in England. Now Robins and D'Alembert had the same conception of a limit. Both held the view that variables cannot reach their limits. However, there was one difference; Robins embodied this restriction in his definition of a limit; D'Alembert omitted it from his definition, but referred to it in his explanatory remarks.

Some of the eighteenth century British conceptions possessed great merit. Perhaps no intuitional conceptions available in the study of the calculus are clearer and sharper than *motion* and *velocity*. These ideas offer even now great help in approaching the first study of the calculus. A second point of merit lay in the abandonment of the use of infinitely small quantities. Not all English authors of the eighteenth century broke away from infinitesimals, but those who did were among the leaders: Robins, Maclaurin, Simpson, Vince, and a few others.

From the standpoint of rigor, the treatment of the calculus by these men was far in advance of the Continental. In Great Britain there was achieved in the eighteenth century in the geometrical treatment of fluxions that which was not achieved in the algebraical treatment until the nineteenth century. It was not until after the time of Weierstrass that infinitesimals were cast aside by mathematical writers on the Continent.

Judged by modern standards all eighteenth century expositions of the calculus, even the best British expositions, are defective. As pointed out by Landen and Woodhouse, there was an unnaturalness in founding the calculus upon *motion* and *velocity*. These notions apply in a real way only to dynamics. Moreover, not all continuous curves can be conceived as traceable by the motion of a point. The notion of variable velocity is encumbered with difficulties. Then again, in all discussion of limits during the eighteenth century, the question of the *existence of a limit* of a given sequence was never raised. The word "quantity" was not defined; quantities were added, subtracted, multiplied and divided. Were these quantities numbers, or were they considered without reference to number? Both methods are possible. Which did British authors follow? No explicit answer to this was given. Our understanding of authors like Maclaurin, Rowe and others, is that in initial discussions such phrases as "fluxion of a curvilinear figure" are used in a non-arithmetical sense; the idea is purely geometrical. When later the finding of the fluxions of terms in the equations of curves is taken up, the arithmetical or algebraical conception is predominant. Rarely does a writer speak of the difference between the two. Perhaps

> "His notions fitted things so well
> That which was which he could not tell."

The theory of irrational number caused no great anxiety to eighteenth century workers. Operations applicable to rational numbers were extended without scruple to a domain of numbers which embraced both rational and irrational. There was no careful exposition of the number system used. The modern theories of irrational number have brought about the last stages of what is called the *arithmetization* of mathematics. As now developed in books which aim at rigor the notion of a limit makes no reference to quantity and is a purely ordinal notion. Of this mode of treatment the eighteenth century had never dreamed.

NOTE ON SOME APPLICATIONS OF A GEOMETRICAL TRANSFORMATION TO CERTAIN SYSTEMS OF SPHERES.[1]

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In a paper in Volume VI of the *Proceedings of the Edinburgh Mathematical Society*, Professor Allardice considers the transformation *in plano:*

[1] Presented to the San Francisco Section of the American Mathematical Society, April 12, 1913.

Let l be a fixed straight line, C any plane curve, t a tangent to C meeting l in X and making an angle θ with l; then C', the transformed curve of C, is the envelope of a straight line t' through X making an angle ϕ with l, where θ and ϕ are connected by the relation

$$\tan \tfrac{1}{2}\phi = k \tan \tfrac{1}{2}\theta.$$

The length of the tangent from any point in l to C remains unaltered by the transformation. A similar method is applicable in space and the following paper gives some of the results in its applications to certain systems of spheres. The method of transformation is as follows:

Let α be a fixed plane, S any surface, and β a plane tangent to S intersecting α in the line i and making with α an angle θ. If β' be a plane through i making with α an angle ϕ, determined by the relation

$$\tan \tfrac{1}{2}\phi = k \tan \tfrac{1}{2}\theta, \tag{1}$$

then the envelope of β' will be defined as the transformed surface of S, or, more simply, the "*transform of S*."

From the given relation, $\tan \frac{1}{2}\phi = k \tan \frac{1}{2}\theta$, and the identity,

$$\tan \phi = \frac{2 \tan \phi/2}{1 - \tan^2 \phi/2},$$

we find that

$$\tan \phi = \frac{2k \tan \theta}{\pm (1 - k^2) \sqrt{1 + \tan^2 \theta} + 1 + k^2}. \tag{2}$$

It is evident from the nature of this relation that for every value of k there are two values of $\tan \phi$, according as we consider the angle $\alpha\beta$ as θ or $(\theta + \pi)$.

For simplicity, we will use Cartesian coördinates to find the equation of the transform of a given sphere from its equation, and we will take the xy-plane as the plane of transformation, α. Let θ be the angle formed by the xy-plane and the plane $ux + vy + wz - 1 = 0$, and ϕ be the angle formed by the xy-plane and the plane $ux + vy + w_1 z - 1 = 0$. (Since the intercepts of both planes on the x-axis and on the y-axis are equal, respectively $1/u$ and $1/v$, they intersect the xy-plane in the same line, as required by the transformation.)

From the formula for the angle between two planes,

$$\tan^2 \delta = \frac{(BC' - B'C)^2 + (CA' - C'A)^2 + (AB' - A'B)^2}{(AA' + BB' + CC')^2},$$

we have

$$\tan^2 \theta = \frac{u^2 + v^2}{w^2} \qquad \text{and} \qquad \tan^2 \phi = \frac{u^2 + v^2}{w_1^2}.$$

Substituting these values in equation (2) and solving for w, we obtain

$$w = \frac{(1 + k^2) w_1 \pm (1 - k^2) \sqrt{u^2 + v^2 + w_1^2}}{2k}, \tag{3}$$

or,

$$(4)\quad w^2 = \frac{2(1+k^4)w_1^2 + (1-k^2)^2(u^2+v^2) \pm 2w_1(1-k^4)\sqrt{u^2+v^2+w_1^2}}{4k^2}.$$

Now, let the center of the sphere lie in the z-axis and its equation will be

$$(5)\quad x^2 + y^2 + z^2 + 2nz + d = 0.$$

Expressing the condition that the plane $ux + vy + wz - 1 = 0$ be tangent to this sphere, we have

$$(6)\quad (d-n^2)u^2 + (d-n^2)v^2 + dw^2 + 2nw + 1 = 0,$$

which is the tangential equation of the sphere.

Substituting the values of w and w^2 found in (3) and (4) and then simplifying and factoring the result, we have the equation of the transform of the given sphere

$$(7)\quad \begin{aligned}&\big[\{(k^2+1)^2d - 2n^2(k^4+1) + 2n(k^4-1)\sqrt{n^2-d}\}(u^2+v^2) + 4dk^2w_1^2\\ &+ 4k^2 + 4k\{n(k^2+1) + (k^2-1)\sqrt{n^2-d}\}w_1\big] \times \big[\{(k^2+1)^2d\\ &- 2n^2(k^4+1) - 2n(k^4-1)\sqrt{n^2-d}\}(u^2+v^2) + 4dk^2w_1^2 + 4k^2\\ &+ 4k\{n(k^2+1) - (k^2-1)\sqrt{n^2-d}\}w_1\big] = 0.\end{aligned}$$

This equation breaks up into two equations which are the tangential equations of two spheres, and may be written in the form

$$(8)\quad A_1(u^2+v^2) + Cw_1^2 + 2N_1w_1 + D = 0;$$

$$(9)\quad A_2(u^2+v^2) + Cw_1^2 + 2N_2w_1 + D = 0.$$

The Cartesian equations of these forms are

$$(10)\quad (CD - N_1^2)x^2 + (CD - N_1^2)y^2 + A_1Dz^2 + A_1C + 2A_1N_1z = 0;$$

$$(11)\quad (CD - N_2^2)x^2 + (CD - N_2^2)y^2 + A_2Dz^2 + A_2C + 2A_2N_2z = 0,$$

where, from the actual values of the coefficients involved,

$$(CD - N_1^2) = A_1D, \qquad \text{and} \qquad (CD - N_2^2) = A_2D.$$

Simplifying and substituting, we have finally

$$(12)\quad 4k^2(x^2+y^2+z^2) + 4k\{n(k^2+1) + (k^2-1)\sqrt{n^2-d}\}z + 4dk^2 = 0;$$

$$(13)\quad 4k^2(x^2+y^2+z^2) + 4k\{n(k^2+1) - (k^2-1)\sqrt{n^2-d}\}z + 4dk^2 = 0:$$

the equations of two spheres whose centers lie in the z-axis. Hence, in general, a sphere is transformed into two spheres whose centers lie in a line perpendicular to the plane of transformation and passing through the center of the given sphere.

Now, let ρ represent the radius of the given sphere and δ the distance of its

center from the plane of transformation; also, let r' and r'' and d' and d'' represent respectively the radii and distances from the plane of transformation of the centers of the transformed spheres. The equations of the latter may then be written

$$x^2 + y^2 + z^2 + [-\delta(k + 1/k) + (k - 1/k)\rho]z + d = 0; \tag{14}$$

$$x^2 + y^2 + z^2 + [-\delta(k + 1/k) - (k - 1/k)\rho]z + d = 0: \tag{15}$$

whence the resulting relations:

$$\begin{aligned} d' + r' &= (\delta - \rho)k; \qquad d'' + r'' = (\delta + \rho)k; \\ d' - r' &= (\delta + \rho)/k; \qquad d'' - r'' = (\delta - \rho)/k. \end{aligned} \tag{16}$$

We will next consider for what values of k the transformed spheres will degenerate into points. It is evident that we must consider each case separately.

In the first case we have

$$r' = \left| -\frac{1}{2k}\{(\delta + \rho) - k^2(\delta - \rho)\} \right| = 0,$$

whence

$$k^2 = (\delta + \rho)/(\delta - \rho). \tag{17}$$

Here we have two real values for k, provided the given sphere does not cut the plane of transformation; and a zero or infinite value for k if the given sphere is tangent to that plane. The corresponding values of the distance are

$$d' = \mp\sqrt{\delta^2 - \rho^2},$$

or in terms of the original coefficients,

$$d' = \mp\sqrt{d}. \tag{18}$$

In the second case, by similar methods, we find that

$$k^2 = (\delta - \rho)/(\delta + \rho), \tag{19}$$

and

$$d'' = \mp\sqrt{\delta^2 - \rho^2}, \quad \text{or,} \quad d'' = \mp\sqrt{d}. \tag{20}$$

It follows that, in general, a sphere may be transformed into a point for any one of four values of k, the points being the same in pairs, and all four being equidistant from the plane of transformation, one pair on each side. It is to be especially noted that while each of the two transformed spheres may degenerate into points, both spheres cannot become points at the same time, unless either $\rho = 0$, or $\delta = 0$; *i. e.*, when the given sphere is itself a point and this point may be transformed into its image with respect to the plane of transformation, or when the center of the given sphere lies in the plane of transformation and the points are imaginary. We may also note that a point can be transformed into only one sphere.

We have already seen, equations (12) and (13), that for the same value of k a given sphere may be transformed into two distinct spheres. These two spheres evidently result from the two values of ϕ, according as we consider the angle $\alpha\beta$ as θ or $(\theta + \pi)$. In case the angle is θ we will say that the transformed sphere is traced out by a *direct* movement, and where the angle is $(\theta + \pi)$ we will say that it is traced out by an *indirect*, or *inverse*, movement. It is evident that if we transform two spheres, *both* directly, or *both* inversely, the *direct* common tangent planes to the two spheres become common tangent planes to the transformed spheres, while if one sphere is transformed directly and the other inversely, the *transverse* common tangent planes become common tangent planes to the transformed spheres.

Forming the equation of the radical plane of the given sphere and either of its transforms, we have

$$z = 0,$$

which is also the plane of transformation. It follows that the plane of transformation is the radical plane of the given sphere and its transforms, and that the length of a tangent line from any point in the plane of transformation to the sphere is unaltered by the transformation. Hence the distance between the points of contact of common tangent planes to two spheres remains unaltered. This is the important property of the transformation. By transforming several spheres into points at the same time, relations and properties of spheres may be obtained directly from known relations between points.

For two or more spheres to be transformed simultaneously into points, it is evidently sufficient that the values of k^2 be equal. Considering the case of two spheres with radii ρ' and ρ'' and with the distances of their centers from the plane of transformation, δ' and δ'', respectively, we have

$$k^2 = (\delta' - \rho')/(\delta' + \rho') \quad \text{and} \quad k^2 = (\delta'' - \rho'')/(\delta'' + \rho'');$$

whence

$$(\delta' - \rho')/(\delta' + \rho') = (\delta'' - \rho'')/(\delta'' + \rho''), \quad \text{or} \quad \delta'/\delta'' = \rho'/\rho''.$$

This is the condition that the plane of transformation pass through the direct center of similitude of the two spheres. Using the reciprocal values for k^2 we obtain the same result. Finally, let

$$k^2 = (\delta' - \rho')/(\delta' + \rho') \quad \text{and} \quad k^2 = (\delta'' + \rho'')/(\delta'' - \rho'');$$

whence

$$\delta'/\delta'' = -\rho'/\rho'':$$

the condition that the plane of transformation pass through the inverse center of similitude of the two spheres. We conclude that two spheres may be transformed simultaneously into points if the plane of transformation contain a center of similitude of the two spheres. Likewise, if the plane of transformation contain any one of the four axes of similitude of three spheres, these spheres will all be transformed into points by the same transformation. Further, if the plane

of transformation be any one of the eight planes of similitude of four spheres,[1] the four spheres may be transformed into points simultaneously; and, finally, any number of spheres which have a common plane of similitude may be so transformed. Two spheres on the same side of the plane of transformation will be transformed, either both directly, or both inversely; while of two on opposite sides, one will be transformed directly, the other inversely.

By giving k every possible value a given sphere may be transformed into an infinite series of spheres which, by virtue of the properties of the plane of transformation as the radical plane of a sphere and its transforms, is cut orthogonally by a system of spheres. Then, since the given sphere is one of the series, if we transform the whole system for the same value of k, it will transform into itself.

It may be noted at this point that the results here obtained by analytic methods may readily be obtained by synthetic methods, and in the sequel they will be interpreted in accordance with the particular problem under consideration. In all problems which follow it will be necessary to associate only those common tangent planes and distances between points of contact of common tangent planes to two spheres which determine the centers of similitude in the planes of similitude used as planes of transformation. Thus we will associate with the plane of similitude passing through the six direct centers of similitude of four spheres the external common tangent planes of each pair of spheres, and only the external common tangent planes of these spheres.

Some Applications of the Method.

1. The condition that four planes passing through a point should be tangent to the same sphere is given by the equation

$$\tan \tfrac{1}{2}\phi' \tan \tfrac{1}{2}\phi'' = \tan \tfrac{1}{2}\theta' \tan \tfrac{1}{2}\theta'',$$

where ϕ', ϕ'', θ', θ'' are the angles formed by the given planes and the plane determined by the lines of intersection of the two pairs of opposite planes.

Transform the sphere into a point with the external diagonal plane as the plane of transformation, and let the angles of the transformed planes with this plane of transformation be ϕ and θ, corresponding respectively to the angles ϕ', ϕ'' and θ', θ''. We then have the following relations:

$$\tan \tfrac{1}{2}\phi = k \tan \tfrac{1}{2}\phi'; \qquad \tan \tfrac{1}{2}\theta = k \tan \tfrac{1}{2}\theta';$$

$$\tan \tfrac{1}{2}(\phi + \pi) = k \tan \tfrac{1}{2}\phi''; \qquad \tan \tfrac{1}{2}(\theta + \pi) = k \tan \tfrac{1}{2}\theta''.$$

Eliminating ϕ, θ, k from these equations, we have the given condition.

[1] The twelve centers of similitude of four spheres lie in sets of six in a plane: the six direct centers lie in a plane; the three direct centers of any three spheres lie in a plane with the three inverse centers not paired with them; and any two direct centers, using each sphere only once, lie in a plane with the four inverse centers not paired with them. Thus there are eight planes of similitude, one containing all the direct centers; four containing three direct and three inverse centers; and three containing two direct and four inverse centers.

2. If five spheres have a common plane of similitude they may be transformed into five points by the same transformation. Therefore the distances between the points of contact of the common tangent planes of each pair of spheres satisfy the relation connecting the ten straight lines joining five points in space.

Let $d_1, d_2, d_3, \cdots d_{10}$ be the distances. Then the relation is given by the determinant[1]

$$\begin{vmatrix} 0 & d_1^2 & d_2^2 & d_3^2 & d_4^2 & 1 \\ d_1^2 & 0 & d_5^2 & d_6^2 & d_7^2 & 1 \\ d_2^2 & d_5^2 & 0 & d_8^2 & d_9^2 & 1 \\ d_3^2 & d_6^2 & d_8^2 & 0 & d_{10}^2 & 1 \\ d_4^2 & d_7^2 & d_9^2 & d_{10}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

3. If, in addition to the conditions of No. 2, the five spheres also touch a sixth sphere, the ten distances will satisfy the additional relation connecting five points lying on the surface of a sphere.

Transforming the five spheres into points as above, they must lie on the surface of the transform of the sixth sphere and will therefore satisfy the additional relation given by the determinant[1]

$$\begin{vmatrix} 0 & d_1^2 & d_2^2 & d_3^2 & d_4^2 \\ d_1^2 & 0 & d_5^2 & d_6^2 & d_7^2 \\ d_2^2 & d_5^2 & 0 & d_8^2 & d_9^2 \\ d_3^2 & d_6^2 & d_8^2 & 0 & d_{10}^2 \\ d_4^2 & d_7^2 & d_9^2 & d_{10}^2 & 0 \end{vmatrix} = 0.$$

4. The radius R of a given sphere S tangent to four given spheres may be found in terms of R, the distance δ of the center of S from the plane containing the six direct centers of similitude of the four spheres, the distances between the points of contact of common tangent planes to pairs of spheres, and the radius and distance from its center to the plane of similitude of any one of the given spheres.

If we transform the four spheres into points by the same transformation, these four points will lie on the transform of S. Hence

$$R' = \frac{1}{2}\sqrt{\frac{(\Sigma aa')[\Pi(-aa' + bb' + cc')]}{\Sigma[a^2a'^2(b^2 + c^2 - a^2 + b'^2 + c'^2 - a'^2)] - (a^2b^2c^2 - \Sigma a^2b'^2c'^2)}},$$

where a, a', etc., are the distances between the points of contact of the common

[1] Cayley, "Collected Mathematical Papers," Volume I, p. 1 et seq.

tangent planes of two pairs of spheres with no sphere in common, and where Σ and π represent cyclo-symmetrical functions only. But

$$R' = \left| \frac{1}{2k} \{\delta(k^2 - 1) \mp (k^2 + 1)R\} \right|,$$

where

$$k^2 = (d_1 - r_1)/(d_1 + r_1), \quad \text{or} \quad k^2 = (d_1 + r_1)/(d_1 - r_1),$$

from equations (17), and (19).

Hence

$$R = | - \delta(r_1/d_1) \pm 2\{(\sqrt{d_1^2 - r_1^2})/d\}R' | .$$

5. The locus of the center of a sphere S such that the distances between the points of contact of the common tangent planes to S and four fixed spheres are equal is a straight line perpendicular to a plane of similitude of the four spheres.

Transforming the fixed spheres into points, the transformed spheres S' will form a system of spheres with a common center; namely, the center of the sphere determined by the four points. Moreover, all the spheres S' with real tangents to the four points lie within this sphere; consequently the distance between the points of contact cannot exceed the radius of this sphere, and hence it is a maximum when S' becomes a point; *i. e.*, when the four fixed spheres and S have a common plane of similitude. The theorem follows directly from the fact that the system S' is a system of concentric spheres. There are eight cases for consideration corresponding to the eight planes of similitude of the four fixed spheres.

6. The centers of the spheres tangent to four given spheres lie in pairs on eight straight lines passing through the radical center and perpendicular to one of the planes of similitude of the four spheres.

This is only a special case of the preceding. It is evident that the radical plane of any pair of the tangent spheres is the plane of similitude, which is perpendicular to the line joining the centers of the two spheres. The results of this theorem can be employed in the construction of the spheres tangent to four given spheres.

7. If S is a sphere tangent to four given spheres, which are tangent to one another, the five spheres have common planes of similitude.

Transforming the four spheres into points, the points will coincide, since each sphere touches one of the others. But the four points must lie on S', the transform of S, and therefore S' must also reduce to a point simultaneously with the other four; hence the result.

8. The locus of the centers of all spheres which touch a given sphere and have a common plane of similitude with it is an ellipsoid of revolution.

Let r be the radius and d be the distance from the common plane of similitude of the center of the given sphere, and let ρ and δ be the radius and distance respectively of the variable sphere. Transform the spheres into points for the same value of k and we have

$$d' = \frac{d}{2}(k + 1/k) - \frac{r}{2}(k - 1/k); \qquad \delta' = \frac{\delta}{2}(k + 1/k) - \frac{\rho}{2}(k - 1/k).$$

But, since all the spheres are tangent to the given sphere, the points into which they transform are coincident, and therefore, $d' = \delta'$. Hence

$$(k - 1/k)(r + \rho) = \left(d + \delta - \frac{4d'}{k + 1/k}\right)(k + 1/k),$$

or, $(r + \rho)/(\delta + c) = e$, where e and c are constants.

Therefore, the locus of the centers of the spheres is the locus of a point, the ratio of whose distances from the center of the given sphere to its distance from a plane parallel to and at the distance c from the common plane of similitude is constant; *i. e.*, an ellipsoid of revolution.

THE ASTROLABE.

By MARCIA LATHAM, Hunter College, New York City.

Among ancient civilized peoples for many centuries the most important instrument used by astronomers, astrologers and surveyors was the astrolabe, or, more properly, the astrolabe planisphere.

The word is derived from two Greek words meaning "to follow the stars," and is therefore applicable in general to any astronomical or astrological instrument. Indeed, the name has been applied to at least three distinct forms. The first of these, better known as the armillary sphere, or *instrumentum armillarum* (instrument of rings), consists of two, three, or more brass circles, representing the circles of the celestial sphere, hinged together in the proper relations, and bearing tubes for sighting the heavenly bodies. An engraving of this may be found on the title page of many an old book on mathematics or astronomy. It was doubtless invented by Hipparchus, and was used by Ptolemy, who described it in the *Almagest*, Book V, Proposition 1. A large wooden ring, used by mariners in the days of Columbus and Vasco da Gama to take the altitude of the sun, was also known as an astrolabe. The most important form, however, was the astrolabe planisphere, or, simply, the planisphere.

The theory of the planisphere was given in the second century of our era by Ptolemy in a treatise on the planisphere, which is not extant in the Greek but survived in the Arabic, from which it was translated by Commandinus.

The Arabs, probably applying the theory of Ptolemy, constructed exceedingly accurate astrolabes. Remains of astrolabes have been found in Assyrian excavations, and the instrument is still of practical value in India and Persia. From Arabia it passed again into Europe, and was used by astronomers and surveyors in Italy, England and other countries until the eighteenth century. Bion wrote a treatise on the astrolabe in 1702, and Boileau mentioned it in a satire written in 1693:

> "Un astrolabe en main, elle a dans sa gouttière
> À suivre Jupiter passé la nuit entière."

The simplest and most available English work on the description and use of the astrolabe is the *Tractatus de Conclusionibus Astrolabii* (*Bred and Mylke for Childeren*) written by Geoffrey Chaucer in 1391 and 1392, in the form of a letter to his "lytle sonne Lowis," then ten years old, who was already familiar with the celestial globe and had expressed a desire for information concerning the astrolabe. Chaucer's work, according to Professor Skeat, was based upon the "Compositio et Operatio Astrolabii," a Latin treatise by Messahallah, an Arabian astronomer of the ninth century.

More careful descriptions are to be found in the geometrical and astronomical treatises by Clavius (1612), Metius (1626 and 1633), Danti (1569), Bruni (1625), Reisch (1503), Gemma Frisius and Deschâles; but the standard authority is Johannes Stoefler (or Stoflerinus), whose *Elucidatio fabricæ ususque astrolabii* was published in 1510. (The copy consulted in preparing this paper was published in Oppenheim in 1524.)

The astrolabe planisphere is essentially the projection of the celestial sphere upon a plane. In most forms the projection is stereographic, that is, the point of sight is at one of the poles of the circle upon whose plane the sphere is projected. Three primitive planes have been used:

1. The plane of the equator, the point of sight being at the south pole. This form is called the "equinoctial astrolabe," and is the one suggested by Ptolemy. It is also known as the "septentrional astrolabe."
2. The plane of the equinoctial colure.
3. The plane of the solstitial colure.

The objection to the equinoctial astrolabe is that the nature of the projection varies with the latitude. To meet this difficulty Gemma Frisius used the second form, which, being independent of the location of the observer, is called the "universal astrolabe," or "Catholicon."

Johannes de Rojas, a Spaniard, projected the sphere orthographically on the plane of the solstitial colure, the result being the "Analemma," which is also a universal astrolabe. There was still the objection that in general the projections of equal arcs of a circle are not equal. To obviate this difficulty de la Hire (1640–1718) used the method of globular projection, on the plane of the meridian. In this method the point of sight is on the axis of the primitive circle, outside the sphere and at a distance from its surface equal to the product of its radius by the sine of forty-five degrees. This was further modified by Parent, but Clavius returned to the form suggested by Ptolemy, and most of his successors have followed his example.

Summarizing, then, we may say that the most important form is the equinoctial or septentrional astrolabe planisphere, suggested by Ptolemy and revived by Clavius, the principle of which is stereographic projection upon the plane of the equinoctial, the point of sight being at the south pole.

The astrolabe was usually made of brass or copper, circular in shape, and from four to seven inches in diameter (some of the Eastern instruments are much larger), and very carefully and beautifully engraved. Chardin tells us that in

Persia the astronomers themselves made them, not entrusting them to ordinary artisans.

The fundamental part is called the *mater* or mother, and is a heavy circular piece with a ring which permits it to assume a vertical position when suspended from the right thumb. One side is the back or *dorsum*, on which rotates the *rule*, a flat bar bearing sights similar to those of a gun. The other side, called the *front* or face, is hollowed out so that any one of several plates, or *tables*, may be fitted into it. Above such a plate lies the *rete* (net), and the *label*, which is simply a pointer; and the whole is fastened in place by a pin or wedge (see Fig. 1).[1]

FIG. 1.

Examining it more closely we find around the rim of the "back" several rings containing respectively the signs of the zodiac, divided into degrees; the months of the year, divided into days; and sometimes the golden letters of the church calendar, or the names of the winds. Fig. 2 is a diagram of the back. *AB* represents the meridian, and *CD* the east and west line, but it should be noted that the observer's right indicates the west and his left the east. Above the line

[1] This astrolabe is in the collection of Professor David Eugene Smith, of Columbia University.

CD are often placed the "curves of the unequal hours." These are circles, each passing through the center of the astrolabe and one of the six points of division of the quadrant, and having its center on the meridian line.

Below *CD* is the "geometric square" (*Quadratus geometricus*), or "square of the shadows," *EGHKL*. *EG*, *GH*, *HK* and *KL* are each divided into a number of equal parts. The vertical sides constitute the "*umbra versa*," which corresponds to the shadows cast on a wall by a horizontal staff fastened to the side of the wall, the altitude of the sun being not more than forty-five degrees. The horizontal part corresponds to shadows cast on the ground by a vertical staff when the sun's altitude is not less than forty-five degrees, and is called the "*umbra recta*." The geometric square is in itself a complete instrument, and was so used.

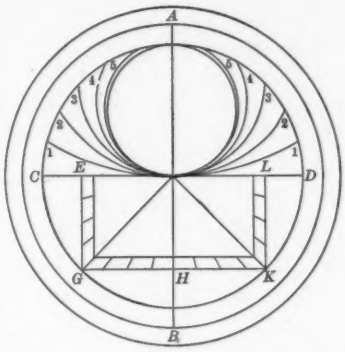


FIG. 2.

Turning the instrument over, we find that the concave part of the face bears three concentric circles, which are the projections of the equator and the tropics, the hole in the center representing the north pole.

The thin plates that fit into this depression constitute the essential part of the astrolabe, and will be described below. The rete, called by the French *l'araigne* (spider) is a plate of filigree metal, consisting of a circle representing one half of the zodiac, and many small tongues of metal, each of which serves to locate some important fixed star. Those within the zodiac have, of course, north latitude, and those without, south latitude. A point projecting from the outer rim of the rete is called the *denticle*. The wedge holding the different parts in place was often in the shape of a horse, and was named the "*equus restringens*."

With the aid of figures 3 and 4 the method of constructing the plates can be easily understood by any one familiar with the principles of stereographic projection.

In figure 3, the largest circle is the projection of the Tropic of Capricorn, *AB* that of the meridian, and *CD* of the east and west line. Arc *AH* is equal to the maximum declination of the sun. *GHM* is the projection of the equator, and *LM* that of the Tropic of Cancer. *AHMN* is therefore that of the ecliptic.

In figure 4 arc *HL* is the complement of the latitude of the place for which the plate is constructed, and *Z* is the projection of the zenith and *Y* that of the nadir. Arc *TK* is equal to arc *HL* and *M* is a point on the oblique horizon. *GMH* is the projection of the principal almucantar, or circle of altitude. Arc *LW* is equal to arc *LX* and circle *EF* is an almucantar. *GHZ* is the projection of the prime vertical. *NP* contains the centers (as *S*) of all azimuths (as *UV*), which must also pass through the zenith.

Below the horizon the plates bear certain "curves of the hours," constructed

as follows: Divide those portions of the tropics and the equator lying below the horizon each into twelve equal arcs, and pass circles through the corresponding points of division.

Lastly, since the instrument was used also for astrological purposes, the plates bear lines dividing the celestial sphere into the twelve astrological "houses." To obtain these, divide the Tropic of Capricorn into twelve equal arcs. Construct a circle through the north point of the horizon, and the points of division respectively nearest the meridian to the right above and to the left below. A similar procedure will determine all the lines separating the twelve houses.

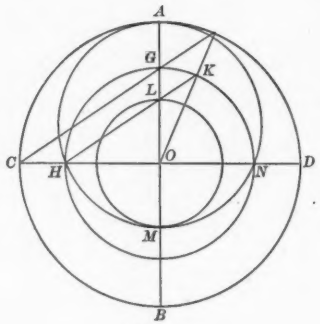


FIG. 3.

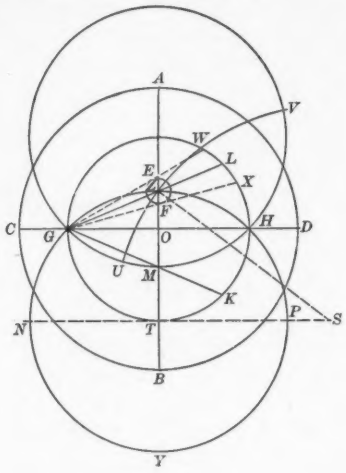


FIG. 4.

Having already explained the construction of the zodiac on the rete, it remains to determine the positions of the tongues locating the fixed stars. The position of a star can easily be fixed when its latitude and longitude are known.

In some cases, particularly among the Orientals, the instrument was constructed for only one latitude; in such instruments the sphere is projected directly upon the mother, and there are no extra plates.

A much longer article than this is to be could well be devoted to an enumeration of the uses of the astrolabe. We shall confine ourselves to mentioning a few of these.

To determine the sign and degree of the zodiac for any day of the year, use the back of the instrument, and lay the label over the day. Its extremity will indicate the corresponding degree.

The back is also used to find the altitude of the sun. Suspend the instrument on the right thumb, and turn the rule until the rays of the sun shine through both sights. The rule will then point out the number of degrees in the sun's altitude. By a similar procedure, the altitude of a star can be found.

The time of the day may be found as follows: First get the sun's altitude; turn the astrolabe over, and revolve the rete (westward or eastward according as the observation is made in the morning or afternoon) until the degree of the zodiac corresponding to that day falls upon the proper circle of altitude. Lay the label across this degree, and it will point out a number of degrees on the margin, from which the time can be calculated.

Other matters as easily determined are the time of dawn and of sunrise for a given day; the meridian altitude of the sun, given the degree of the sun; and, conversely, the degree, when the meridian altitude is known; the latitude of any place; and the cardinal points of the compass.

The curves of the unequal hours on the back may also be used to find the time of the day. Determine the meridian altitude of the sun. Place the rule on the corresponding degree and mark with ink on the rule the point where it cuts the arc of the twelfth hour. Then turn the rule until the sun shines through the sights, and the point marked on the rule will fall on or near the circle corresponding to the hour of the day.

The geometric square was invaluable to the surveyor before the invention of the telescope. The instrument known as "Jacob's Staff" was its only rival in usefulness. By means of it the surveyor could find the height of an accessible or inaccessible tower, by one or two observations; the height of a tower on an inaccessible hill; the distance between two inaccessible points; the distance between two towers not standing on the same plane; the distance of a tower whose height is known; the breadth of a stream; the depth of a pit, or of a valley with sloping walls.

Let it be required, for example, to find the height of a tower when the observer may take any position in front of it. Holding the astrolabe suspended from his right thumb, with the rule fixed in position along the diagonal of the square, the observer will walk backward or forward until he sights the top of the tower along the rule. Then the height of the tower is equal to the distance of the observer from its foot, increased by the height of the observer.

If, however, the observer must not change his position, he may move the rule around until the top of the tower is sighted, and note the point where the rule cuts the side of the square (see Fig. 5). If each side is divided into one hundred equal parts, and the rule cuts off sixty parts on the umbra versa, the height of the tower is equal to sixty hundredths of the distance of the observer from its foot (always increased by the height of the observer). If it cuts off sixty parts on the umbra recta the height is one hundred sixtieths of the distance to its foot. These statements are easily proved by means of similar triangles.

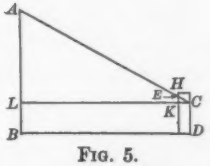


FIG. 5.

If the tower is inaccessible, two observations may be made and the ratio of the height to the distance between the two positions determined. These are some of the simplest applications, but the mathematical principles are the same throughout.

While this article does not give an exhaustive description of the construction and uses of the astrolabe, it is hoped that it may at least serve to show by what means the sciences of astronomy and surveying attained so great a degree of development before the days of Tycho Brahe and Galileo.

A USEFUL PRINCIPLE IN CURVE TRACING.

By ARNOLD EMCH, University of Illinois.

1. The usual elementary methods of curve tracing in rectangular coördinates consist:

(1) In plotting points of the curve by assigning arbitrary values to one of the variables and finding the corresponding values of the other variable;
(2) In determining the tangents at these points by means of the derivatives;
(3) In finding the intersections of the curve with definite straight lines, or other conveniently chosen curves;
(4) In establishing possible properties of symmetry;
(5) In determining concavity and convexity, maxima and minima;
(6) In determining the asymptotes;
(7) In determining possible singular and inflexional points.

This list of tests and different steps to be taken in the investigation of a curve may, of course, be extended according to the difficulties and the nature of the problem.[1]

In ordinary curve tracing little use is made of other algebraic methods than those mentioned above. It is the purpose of this note to show the effectiveness of a certain algebra-geometric method in elementary curve tracing.

2. Restricting ourselves to algebraic curves, let such a curve be represented by the equation

$$F(x, y) = 0 \tag{1}$$

in which $F(x, y)$ is an irreducible polynomial. It is always possible to write $F(x, y) = 0$ in the identical form

$$F(x, y) \equiv \phi_1(x, y)\psi_2(x, y) - \phi_2(x, y)\psi_1(x, y) = 0, \tag{2}$$

where ϕ_1, ϕ_2, ψ_1, ψ_2 are also polynomials in x and y, of which some may reduce to constants. If it should prove to be convenient, we might use

$$F(x, y) - G(x, y) + G(x, y)$$

in place of $F(x, y)$. Thus, as $G(x, y)$ may be any polynomial, we may resolve $F(x, y)$ in an infinite number of ways into the form (2). Equation (2) is evidently the result of the elimination of λ between the simultaneous equations

[1] See PERCIVAL FROST: *An Elementary Treatise on Curve Tracing*, pp. 177–187 (1911).

$$\phi_1(x, y) + \lambda\phi_2(x, y) = 0, \tag{3}$$

$$\psi_1(x, y) + \lambda\psi_2(x, y) = 0. \tag{4}$$

All sets of points (x, y) that satisfy (3) and (4) simultaneously also satisfy (2). Conversely, any set (x_1, y_1) that satisfies (2), with the possible exception of a limited number of sets that satisfy (2) for some particular value of λ, is a simultaneous solution of (3) and (4). As a matter of fact

$$\frac{\phi_1(x_1, y_1)}{\phi_2(x_1, y_1)} = \frac{\psi_1(x_1, y_1)}{\psi_2(x_1, y_1)} = -\lambda_1,$$

for all sets (x_1, y_1) of (2), for which these ratios exist. For a variable parameter λ, (3) and (4) represent two projective pencils of curves, whose product is the given curve (2). Hence the well-known

THEOREM. *An algebraic curve may always be generated by two projective pencils of algebraic curves.*

This theorem forms the basis of Steiner's method of investigation of algebraic curves and is closely connected with Noether's famous theorem concerning the possibility of representing $F(x, y)$ in the form given in (2), when ψ_1 and ψ_2 are given.[1]

3. The possibility of representing a polynomial in the form given in (2) and by Steiner's theorem may be utilized with great advantage in cases where the explicit representation of one variable in terms of the other involves the solution of an equation of higher than the second or third degree. But the method may also be used in any other case and may be embodied in the following principle:

Resolve $F(x, y)$ into the identical form $\phi_1(x, y)\psi_2(x, y) - \phi_2(x, y)\psi_1(x, y)$, in such a manner that the equations

$$\phi_1(x, y) + \lambda\phi_2(x, y) = 0,$$

$$\psi_1(x, y) + \lambda\psi_2(x, y) = 0,$$

assume as simple forms as possible (geometrically). Construct these curves for as many values of λ as seem necessary; then for every value of λ the two curves intersect in points of the given curve.

4. Examples:

Conics. The application of the principle to equations of the second degree leads to the well-known projective theory of conics. ϕ_1, ϕ_2, ψ_1, ψ_2 are linear in x and y, and (3) and (4) become ordinary projective pencils of straight lines.

A quartic: $x^4 - y^4 - x^2y^2 + x^3y - x^3 + y^2 = 0$. This may be written in the form

$$x^3(x + y - 1) - y^2(x^2 + y^2 - 1) = 0, \tag{5}$$

[1] See references to this subject in Pascal's *Repertorium der höheren Mathematik*, Vol. II, first part, 2d ed. (1910), pp. 287–289 and pp. 306–307.

so that the quartic may be generated by the two projective pencils,

$$x^2 + y^2 - 1 - 2\lambda(x + y - 1) = 0, \tag{6}$$

$$-x^3 + 2\lambda y^2 = 0. \tag{7}$$

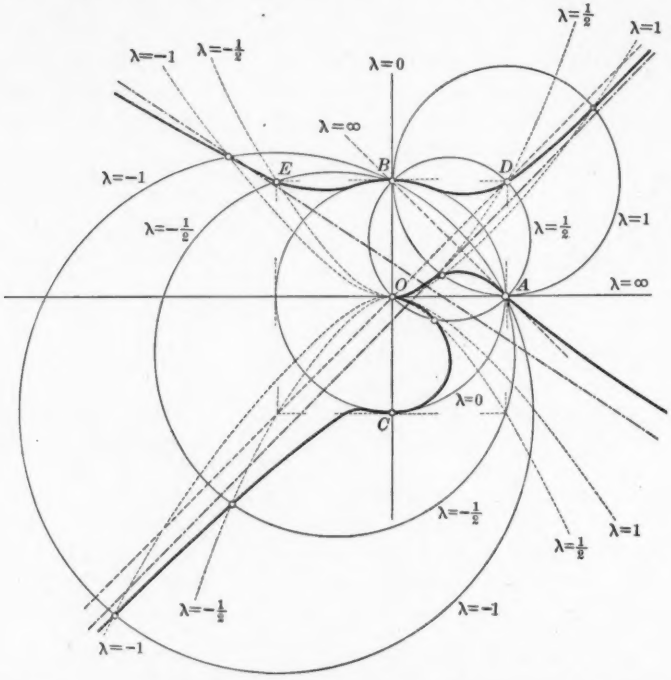
Equation (6) may be written in the form

$$(x - \lambda)^2 + (y - \lambda)^2 = (\lambda - 1)^2 + \lambda^2$$

and represents a circle whose center has the coördinates (λ, λ), and whose radius is $\sqrt{(\lambda - 1)^2 + \lambda^2}$, and which passes through the points $A(1, 0)$ and $B(0, 1)$ (see figure.) Hence, (6) represents a pencil of circles through A and B, with their centers on the bisector of the first and third quadrants. Writing (7) in the form

$$y^2 = \frac{x^3}{2\lambda}, \tag{9}$$

a semi-cubical parabola is obtained, which passes through the points $(2\lambda, \pm 2\lambda)$, which has a cusp at the origin with the x-axis as a tangent, and which may easily



be plotted for a given value of λ. In fact, when μ is any real number, all points $(2\lambda\mu^2, \pm 2\lambda\mu^3)$ lie on (9). For $\lambda > 0$ the cubics lie on the right side of the y-axis;

for $\lambda < 0$ on the left side. For $\lambda = \frac{1}{2}$ the circle (6) passes through the origin and cuts the corresponding cubic $y^2 = x^3$ in two coincident points. Hence, as for this value of λ the circle is not tangent to the cubic at the origin, the quartic must have a singular point at the origin. For $\lambda = \frac{1}{2} - \epsilon$ (ϵ = arbitrarily small positive quantity) the circle does not cut the cubic in real points in the neighborhood of the origin. For $\lambda = \frac{1}{2} + \epsilon$ there are two real points of intersection arbitrarily close to the origin on the right side of the y-axis, one below, the other above the x-axis. From this we conclude that the singularity of the quartic at the origin is a cusp. When $\lambda \doteq 0$, (6) and (7) approach $x^2 + y^2 - 1 = 0$ and $x^3 = 0$, which shows that at B and C (see figure) the circle has three points in common with the quartic. In other words, the unit-circle osculates the quartic at B and C. But at B we have an additional point of intersection of the unit-circle with the line $x + y - 1 = 0$, so that at B the unit-circle has a contact of the third degree with the quartic.

When $\lambda \doteq \infty$, (6) and (7) approach $x + y - 1 = 0$ and $y^2 = 0$, which shows that the quartic touches the line $x + y - 1 = 0$ at A. But another point of intersection, that of the same line with $x^2 + y^2 - 1 = 0$, at A, together with the other two, makes $x + y - 1 = 0$ an inflexional tangent at A. It is easily seen that the quartic also passes through $D(1, 1)$ and $E(-1, 1)$. Any number of other points of the quartic may be obtained as intersections of corresponding curves of the two projective pencils (6) and (7). In the figure, such intersections are shown for $\lambda = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \infty$. The two real asymptotes were determined by the method of substitution.

Lemniscate.[1] The equation of this bicircular rational quartic is

$$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) = 0, \tag{10}$$

and may be written in the form

$$(x^2 + y^2)(x^2 + y^2) - 2a^2(x + y)(x - y) = 0. \tag{11}$$

It is the product of the two projective pencils of circles

$$(x^2 + y^2) + \lambda 2a(x + y) = 0, \tag{12}$$

$$a(x - y) + \lambda(x^2 + y^2) = 0. \tag{13}$$

These equations may be written in the form

$$(x + \lambda a)^2 + (y + \lambda a)^2 = 2\lambda^2 a^2, \tag{14}$$

$$\left(x + \frac{a}{2\lambda}\right)^2 + \left(y - \frac{a}{2\lambda}\right)^2 = \frac{a^2}{2\lambda^2}. \tag{15}$$

They represent two projective pencils of circles which all pass through the origin. All circles of (14) are tangent to the line $x - y = 0$; all circles of (15) are tangent

[1] The name of this curve is derived from the Greek $\lambda\eta\mu\nu\iota\sigma\chi o\varsigma$, which means a loop in shape of the figure 8, and dates back to Jac. Bernoulli who discovered this curve. See his works (Genevae, 1744), Vol. 1, p. 609.

to the line $x + y = 0$. The centers of the circles (14) lie on $x + y = 0$, those of (15) on $x - y = 0$. The construction of the circles (14) and (15) for a given λ is therefore extremely simple, and consequently also the construction of points of the lemniscate as intersections of (14) and (15). It is obvious that the circular points at infinity (isotropic points) and the origin are singular points (inflexional double points) of the quartic.

In an equally simple manner the theorem can be established, by the same principles, that all bicircular quartics may be generated by projective pencils of circles.

It is hoped, however, that the few examples explained above will suffice to show the great value of the principle for actual constructive purposes.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

A Short Course in Elementary Mechanics for Engineers. By CLIFFORD NEWTON MILLS, B.S., A.M., Assistant Professor of Mathematics, South Dakota State College of Agriculture and Mechanic Arts. The D. Van Nostrand Company, New York, 1916. xi + 127 pages. $1.00.

This little book contains in a condensed form a discussion of the simpler subjects in kinematics, kinetics and statics. As only uniform motion, uniformly accelerated motion, and the forces which produce such motions are considered, the author has willed to treat them without the use of mathematics beyond trigonometry. The course is written from the definition viewpoint, much detailed discussion being omitted. For this reason and for the further reason that the clean-cut demonstrations of the calculus are not used the reasoning may not be always as rigid as the old-time mathematicians would like.

The engineering unit, the pound, is used throughout as the unit of force; it is defined as "that force which will accelerate a mass of one pound g feet per second per second." The pound of mass is not definitely defined but these relations are given,

$$\text{Weight} = \text{mass} \times g$$

$$\text{Mass} = W/g$$

thus avoiding hair-splitting distinctions.

Problems involving center of gravity and moment of inertia have been ingeniously worked out without the calculus; true, the summation sign, Σ, has been employed, but the summations are evaluated by algebraic processes.

Friction which might have been treated in an elementary manner, and which is of great importance to machinists and others who would have use for this book, seems to have been omitted. There is a large supply of well-graded problems.

The reviewer does not wish to give his full endorsement to a work in mechanics

for "engineers" that does not use the calculus. It was well enough in the early days to "work up" in the profession, to apply from a hand-book formulas about the derivation of which he was ignorant. But at the present time engineering is supposed to be a learned profession, and the engineer who is unacquainted with calculus can hardly be said to be up to date. But on the whole this book ought to prove an excellent work for beginning classes in mechanics, for extension courses, for trade and manual training schools, and for those who do not understand the calculus or prefer not to use it.

The following correction is noted at the suggestion of the author. The line near the middle of page 55 should read as follows: "Now, if the values of W, a, b, and c are known, and $a = b$, then".

GEORGE R. CHATBURN.

UNIVERSITY OF NEBRASKA.

Elements of Analytic Geometry. By A. ZIWET, Professor of Mathematics, University of Michigan, and A. D. HOPKINS, Instructor in Mathematics, University of Michigan. New York, The Macmillan Company, 1916. vi + 280 pages.

The present edition of this Analytic Geometry is a reproduction of the earlier book by the same authors[1] with the omission of those parts concerned chiefly with college algebra. The reason given for the abridged edition is that college algebra is usually taught previously to, and independent of analytic geometry.

Thus, in discussing the intersection of two straight lines, the determinant form of the solution is retained, but the development of the determinant is omitted.

The algebraic treatment of simultaneous equations, determinants, permutations and combinations, complex numbers and theory of algebraic equations, including numerical equations is omitted.

The part on the plane now occupies 180 pages. The part on three dimensions is almost exactly as before, and occupies 75 pages. As the parts retained have not been altered, the comments made on the previous book apply to this one.

VIRGIL SNYDER.

CORNELL UNIVERSITY,
ITHACA, N. Y.

Differential and Integral Calculus. By CLYDE E. LOVE. The Macmillan Co., New York, 1916. xviii + 343 pages. $2.10.

The first impression made by a book is physical; with this text that impression is most agreeable. The 339 5 x 8 inch pages of excellent typography give the impression of skillful brevity. A careful examination shows that none of the topics of the traditional American text is omitted and that after all the book is of about the usual length. In fact the preface lays no claim to brevity or any other sort of novelty. It is clearly a book well tested before publication, bearing

[1] *Analytic Geometry and Principles of Algebra*, 1913. Reviewed in this MONTHLY, Vol. xxi, pp. 85–89.

no marks of an author's idiosyncrasies. There is no suggestion of anything but a standard treatment of a well-delimited mathematical subject. No applications outside of geometry and mechanics appear, and since in these fields the problems deal extensively with interesting but unnatural curves and with material points having mass but no volume it is questionable whether an ordinary student will discover any applications at all.

The topics are well arranged. Differentiation has numerous applications before transcendental functions are taken up. A moderate treatment of integration (culminating in integration by parts and rational fractions) is followed by all sorts of problems in single integration except fluid pressure. There is then a return to the differential calculus for Taylor's Series and partial differentiation. Then come multiple integrals, with fluid pressure in a chapter by itself. A good introduction to differential equations, some kinematical applications, and an index close the book.

Under maxima and minima (p. 41) problems 26–32 seem to be new, and there is a bit of graphical differentiation on p. 43 which is out of the ordinary. But in the main the standard problems and the standard methods of varying them have been used.

Some features which the reviewer enjoyed were: the arrow notation for limits, the avoidance of the too explicit formulas for summation (for instance $\pi \int r^2 \cdot dh$ is given for a volume of revolution rather than $\pi \int y^2 \cdot dx$, exceptional cases given among the problems on the too obvious Rolle's Theorem, the use of the steel spring instead of a sinusoidal formula in illustrating a type of limit, and the honesty of the admission of the reason for taking up triple integrals. The use of the phrases "arbitrarily close approximation" (p. 151), "expand about the point $x = a$" (p. 223), "cross derivatives" (p. 238) is to be commended. The reviewer hopes that Osgood's adjective "respectable" as applied to functions, and Kowaleski's "almost all" ("fast alle") may next be admitted to the dignified currency which such a textbook confirms.

De minimis non curat lex, but trifles furnish much of his opportunity to the reviewer.

Fault may be found with the statement (p. 51) about the meaning of arc sin x, according to which we must infer that the area under $y^2(1 - x^2)$ from $x = 0$ to $x = 1$ is the *angle* $\pi/2$ radians. Surely in the calculus arc sin means a *number* (of radians in the angle), not the *angle* itself (however measured).

On page 59 the important fact that $e = 2.718 +$ is predicted but it is justified only in an exercise to be solved by the student later (on page 230). On page 228 the limiting behavior of the important function $x^n/n!$ is obscurely disposed of by reference to an exercise seven pages earlier.

No reason is given for adopting the notation dy/dx (p. 14) nor is the student warned against the false interpretation of dy and dx which he inevitably adopts here. Again no reason is assigned for introducing differentials (p. 68) in chapter 6; no use is made of them until chapter 11 and then only on the plea (p. 116) that reasons will appear later. As a matter of fact the reasons do appear on

pages 122 and 150, and the reviewer thinks that there should be this definite reference forward on pages 68 and 116.

All the expansions are to infinite series; the familiar approximations for reciprocals, logarithms, roots, etc., so useful in computing, are nowhere given.

In dealing with variables, limits, and infinitesimals, there is nowhere shown an actual variable in the modern sense, a succession of numerical values; the ordinary student thinks that constants enjoy a monopoly of the Arabic notation, variables requiring always to be *spelled* in letters.

Perry says, in his *Calculus for Engineers*, that the only integrals needed outside of pure theory are $\int x^n \cdot dx$, $\int dx/x$, and $\int e^x \cdot \sin x \cdot dx$. The latter does not appear in this text. Hyperbolic functions appear on page 64 and disappear on page 65, leaving no applications behind.

The proof (p. 123) that we may make a substitution before integrating will seem unnecessary to students just as it does later (p. 267 and p. 272) to the author himself.

"A feature of the book is its insistence on the importance of checking the results of exercises." In every respect but this the text seems to bear out the claims set up in the preface. The reviewer found no attention given to the general question of checking, and except under double integration (where more than one order of work is often possible) very few problems call for solutions in two ways. Moreover, the book omits a great many answers in places where a printed answer is the only reliable check a student can get. No rough methods of checking are suggested, such as sketching derivative or integral curves or other graphical devices, nor is there any hint of checking limits by computing neighboring values, nor of checking differentials by computing small increments, nor of checking integrals by Simpson's rule.

A few criticisms which might equally well be made of other texts beside the one under review are: the absence of any but abstract exercises in indeterminate forms, the failure to illustrate the important formula

$$\frac{df(x, y)}{dv} = \frac{\partial f}{\partial x}\frac{dx}{dv} + \frac{\partial f}{\partial y}\frac{dy}{dv},$$

so that the student can realize that it is good for anything or even recognize it when he meets it in physics, the retention of the phrase "total pressure" for the force on a plane area, and the pretense that *infinite series* are used in computation. The reviewer contends that it is approximation formulas that are used in computation, infinite series being used only in analysis.

W. R. RANSOM.

TUFTS COLLEGE, MASS.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

ALGEBRA.

479. Proposed by S. A. COREY, Albia, Iowa.

Prove or disprove

$$\left\{\begin{vmatrix} x & -v & -z \\ -y & -z & v \\ -z & y & -x \end{vmatrix}^2 + \begin{vmatrix} y & -v & -z \\ x & -z & v \\ v & y & -x \end{vmatrix}^2 + \begin{vmatrix} x & y & -z \\ -y & x & v \\ -z & v & -x \end{vmatrix}^2 + \begin{vmatrix} x & -v & y \\ -y & -z & x \\ -z & y & v \end{vmatrix}^2\right\} \div \begin{vmatrix} x & -y & -z & v \\ y & x & -v & -z \\ z & v & x & y \\ v & -z & y & -x \end{vmatrix}^2 = (x^2+y^2+z^2+v^2)^{-1}.$$

480. Proposed by FRANK IRWIN, University of California.

Solve the equation

$$(x-1) - 2\left(1-\frac{1}{x}\right) - 3\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right) - 4\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)\left(1-\frac{3}{x}\right) - \cdots - n\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)\cdots\left(1-\frac{n-1}{x}\right) = 0.$$

Also the equation

$$(x-a_1) - a_2\left(1-\frac{a_1}{x}\right) - a_3\left(1-\frac{a_1}{x}\right)\left(1-\frac{a_2}{x}\right) - \cdots - a_n\left(1-\frac{a_1}{x}\right)\left(1-\frac{a_2}{x}\right)\cdots\left(1-\frac{a_{n-1}}{x}\right) = 0.$$

[Adapted from a formula of Tait's.]

GEOMETRY.

512. Proposed by J. L. RILEY, Northeastern State Normal School, Tahlequah, Okla.

Determine, geometrically, where the circle of curvature at any point of an ellipse again meets the ellipse.

513. Proposed by ALBERT A. BENNETT, University of Texas.

The following construction for angle-trisection was given some years ago in a non-mathematical journal. Let ABC be a right triangle with AB as hypothenuse. Let BD be a ray drawn parallel to AC and extending in the same direction. Let AEF be a variable ray meeting the segment BC in E, and the ray BD in F. Show, by elementary methods, that when the variable ray is so adjusted that $EF = 2AD$, then $\angle EAC = \frac{1}{3} \angle BAC$.

CALCULUS.

427. Proposed by ROGER S. JOHNSON, Adelbert College, Cleveland, Ohio.

Of all ellipses circumscribed about a given parallelogram, the maximum, with regard to area, has as conjugate diameters the diagonals of the parallelogram.

428. Proposed by J. L. RILEY, Northeastern State Normal School, Tahlequah, Okla.

A loop of a lemniscate rolls in contact with the axis of x. Prove that the locus of the node is given by the equation

$$1 + \frac{dy}{dx} = \left(\frac{a}{y}\right)^{\frac{1}{3}}$$

and that $2\rho\rho' = a^2$, if ρ, ρ' be corresponding radii of curvature of this locus and the lemniscate.

MECHANICS.

344. Proposed by J. ROSENBAUM, New Haven, Conn.

Two bodies of equal masses, and coefficients of friction μ_1 and μ_2 are connected by a light, flexible string, and placed on an inclined plane. What is the angle, θ, between the string and the plane if the inclination, α, of the plane is a minimum when the bodies are on the point of motion?

345. Proposed by J. L. RILEY, Northeastern State Normal School, Tahlequah, Okla.

Two particles A and B are together in a smooth circular tube. A attracts B with a force whose acceleration is ω^2 and moves along the tube with uniform angular velocity 2ω, B being initially at rest: prove that the angle Φ subtended by AB at the center after a time t is given by the equation

$$\log \tan \frac{\pi + \Phi}{4} = \omega t.$$

NUMBER THEORY.

263. Proposed by J. L. RILEY, Northeastern State Normal School, Tahlequah, Okla.

To find values, positive integral, which verify the equation

$$X^3 + 2 = Y^2 \qquad \text{(Gerono).}$$

264. Proposed by C. F. GUMMER, Kingston, Ontario.

Find a general formula for three squares in arithmetical progression. Is it possible for the common difference to be a perfect square?

SOLUTIONS OF PROBLEMS.

ALGEBRA.

A solution of 464, by Mrs. ELIZABETH BROWN DAVIS, was received after selections had been made for publication.

455. Proposed by JOS. B. REYNOLDS, Lehigh University.

Solve for x_n (not in determinant form) the simultaneous equations,

$$\tfrac{4}{3}x_n + 2x_{n-1} + 2x_{n-2} \cdots 2x_4 + 2x_3 + 2x_2 + 2x_1 = g,$$

$$\tfrac{1}{3}\tfrac{0}{3}x_n + \tfrac{2}{3}\tfrac{2}{3}x_{n-1} + 8x_{n-2} \cdots 8x_4 + 8x_3 + 8x_2 + 8x_1 = 4g,$$

$$\tfrac{1}{3}\tfrac{6}{3}x_n + \tfrac{4}{3}\tfrac{0}{3}x_{n-1} + \tfrac{5}{3}\tfrac{2}{3}x_{n-2} \cdots 18x_4 + 18x_3 + 18x_2 + 18x_1 = 9g,$$

$$\tfrac{2}{3}\tfrac{8}{3}x_n + \tfrac{5}{3}\tfrac{2}{3}x_{n-1} + \tfrac{8}{3}\tfrac{8}{3}x_{n-2} + \tfrac{9}{3}\tfrac{4}{3}x_{n-3} \cdots 32x_4 + 32x_3 + 32x_2 + 32x_1 = 16g,$$

$$\tfrac{2}{3}\tfrac{8}{3}x_n + \tfrac{7}{3}\tfrac{6}{3}x_{n-1} + \tfrac{1}{3}\tfrac{1}{3}\tfrac{2}{3}x_{n-2} + \tfrac{1}{3}\tfrac{3}{3}\tfrac{6}{3}x_{n-3} + \tfrac{1}{3}\tfrac{4}{3}\tfrac{2}{3}x_{n-4} \cdots = 25g,$$

. . . . . .

$$(\tfrac{1}{3} + 2n - 1)x_n + (\tfrac{1}{3} + 6n - 5)x_{n-1} + (\tfrac{1}{3} + 10n - 13)x_{n-2} + (\tfrac{1}{3} + 14n - 25)x_{n-3}$$
$$\cdots (\tfrac{1}{3} + 2n^2 - 1)x_1 = n^3 g.$$

SOLUTION BY C. F. GUMMER, Kingston, Ont.

The equations may be written

$$2x_n + 3x_{n-1} + 3x_{n-2} + \cdots + 3x_1 = \frac{3g}{2},$$

$$5x_n + 11x_{n-1} + 12x_{n-2} + \cdots + 12x_1 = \frac{12g}{2},$$

$$8x_n + 20x_{n-1} + 26x_{n-2} + 27x_{n-3} + \cdots + 27x_1 = \frac{27g}{2},$$

. . . . . .

$$(3n-1)x_n + (9n-7)x_{n-1} + \cdots + (3n^2-1)x_1 = \frac{3n^2 g}{2},$$

the coefficients in each column on the left being in arithmetic progression from the principal diagonal downwards, and the differences of the progressions in adjacent columns forming the progression 3, 9, 15, $\cdots$. Let us write the determinant solution for x_n in the form $x_n = d_n'/d_n$.

Then

$$d_n = \begin{vmatrix} 2 & 3 & 3 & \cdots & 3 \\ 5 & 11 & 12 & \cdots & 12 \\ 8 & 20 & 26 & \cdots & 27 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

which becomes, if we subtract adjacent rows, leaving the first unchanged.

$$\begin{vmatrix} 2 & 3 & 3 & 3 & \cdots & 3 \\ 3 & 8 & 9 & 9 & \cdots & 9 \\ 3 & 9 & 14 & 15 & \cdots & 15 \\ 3 & 9 & 15 & 20 & \cdots & 21 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

and on repeating the process

$$\begin{vmatrix} 2 & 3 & 3 & 3 & \cdots & 3 \\ 1 & 5 & 6 & 6 & \cdots & 6 \\ 0 & 1 & 5 & 6 & \cdots & 6 \\ 0 & 0 & 1 & 5 & \cdots & 6 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

If now we perform the same operation once with columns, we get

$$d_n = \begin{vmatrix} 2 & 1 & 0 & 0 & \cdot & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdot & 0 & 0 \\ 0 & 1 & 4 & 1 & \cdot & 0 & 0 \\ 0 & 0 & 1 & 4 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdot & 1 & 4 \end{vmatrix}.$$

Let d_{n-1}, d_{n-2} be corresponding determinants of orders $n-1$ and $n-2$. Developing d_n by means of the last column and row, we obtain $d_n = 4d_{n-1} - d_{n-2}$. Hence, by the usual method for such difference equations,

$$d_n = A(2+\sqrt{3})^n + B(2-\sqrt{3})^n.$$

But $d_1 = 2$ and $d_2 = 7$. Hence $A = B = \frac{1}{2}$, and d_n is given.

Again,

$$d_n' = \frac{g}{2}\begin{vmatrix} 3 & 3 & 3 & 3 & \cdot\cdot & 3 \\ 12 & 11 & 12 & 12 & \cdot\cdot & 12 \\ 27 & 20 & 26 & 27 & \cdot\cdot & 27 \\ 48 & 29 & 41 & 47 & \cdot\cdot & 48 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3n^2 & 9n-7 & \cdot & \cdot & \cdot\cdot & 3n^2-1 \end{vmatrix}.$$

Subtracting the first column from each of the others, we have

$$d_n' = \frac{g}{2}\begin{vmatrix} 3 & 0 & 0 & 0 & \cdot\cdot & 0 \\ 12 & -1 & 0 & 0 & \cdot\cdot & 0 \\ 27 & -7 & -1 & 0 & \cdot\cdot & 0 \\ 48 & -19 & -7 & -1 & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3n^2 & \cdot & \cdot & \cdot & \cdot\cdot & -1 \end{vmatrix} = (-1)^{n-1}\frac{3g}{2}.$$

Hence,

$$x_n = \frac{(-1)^{n-1}3g}{(2+\sqrt{3})^n + (2-\sqrt{3})^n}.$$

467. Proposed by IRA M. DE LONG, University of Colorado.

Determine the function f, from the functional relation $f(x+y) = f(x) + f(y) + 2xy$.

I. Solution by A. Cohen, Johns Hopkins University.

From $f(x+y) = f(x) + f(y) + 2xy$ we have, when $y = 0$, $f(x) = f(x) + f(0)$; hence,

(I) $$f(0) = 0.$$

Similarly, letting $y = x, 2x, 3x$ in turn, we have respectively,

$$f(2x) = 2f(x) + 2x^2; \quad f(3x) = 3f(x) + 6x^2; \quad f(4x) = 4f(x) + 12x^2.$$

This suggests the law,

(II) $$f(nx) = nf(x) + n(n-1)x^2.$$

Assuming this and letting $y = nx$, we have $f[(n+1)x] = (n+1)f(x) + n(n+1)x^2$.

Thus the law is established for n a positive integer.

That it also holds for m/n, a rational fraction, may be seen as follows:

Letting $y = -x$ and using (I)

(III) $$f(-x) = -f(x) + 2x^2.$$

From this readily follows

(III′) $$f(-nx) = -nf(x) + n(n+1)x^2 = -nf(x) + (-n)(-n-1)x^2.$$

If $$y = -\frac{(n-1)}{n}x, \qquad f\left(\frac{x}{n}\right) = f(x) + f\left(-\frac{(n-1)}{n}x\right) - \frac{2(n-1)}{n}x^2.$$

Using (III′), $$f\left(-\frac{(n-1)}{n}x\right) = (1-n)f\left(\frac{x}{n}\right) + \frac{n(n-1)}{n^2}x^2.$$

Hence $$f\left(\frac{x}{n}\right) = f(x) + (1-n)f\left(\frac{x}{n}\right) + \frac{1-n}{n}x^2.$$

From this follows at once $$f\left(\frac{x}{n}\right) = \frac{1}{n}f(x) + \frac{1}{n}\left(\frac{1}{n} - 1\right)x^2.$$

Using (II) and then (IV),

$$f\left(\frac{m}{n}x\right) = mf\left(\frac{x}{n}\right) + \frac{m(m-1)}{n^2}x^2$$

$$= \frac{m}{n}f(x) + \frac{m}{n}\frac{1-n}{n}x^2 + \frac{m}{n}\frac{m-1}{n}x^2 = \frac{m}{n}f(x) + \frac{m}{n}\left(\frac{m}{n} - 1\right)x^2$$

which proves the law (II) for n any positive rational fraction.

Making use of (III′), the law is readily seen to hold for n a negative rational fraction.

Assuming $f(x)$ to be a continuous function of x, the law will hold for n any real number, as can be shown by using a sequence of rational numbers whose limit is n.

Once this is established, the rest follows immediately.

Assuming that $f(x)$ has a derivative, we have on differentiating (II) with respect to n, $x\frac{df(nx)}{d(nx)} = f(x) + (2n-1)x^2$, which holds for all real values of n. In particular, for $n = 1$, $x\frac{df(x)}{dx} = f(x) + x^2$, or $xdf - fdx = x^2dx$. An obvious integrating factor is $1/x^2$. Introducing this and integrating we have $f/x = x + c$; whence

(V) $$f(x) = x^2 + cx, \text{ where } c \text{ is any constant.}$$

II. Solution by Nathan Altshiller, University of Oklahoma.

Let us determine f from the relation

(f) $$f(x+y) = f(x) + f(y) + 2cxy,$$

where c is a given constant.

If in (f) we put $y = x$, we have

(1) $$f(2x) = 2f(x) + 2cx^2$$

and if we put $y = 2x$, we have

$$f(x+2x) = f(x) + f(2x) + 2c \cdot x \cdot 2x,$$

which may be written, taking (1) into account,

$$f(3x) = 3f(x) + 3\cdot 2cx^2. \tag{2}$$

In order to prove that this last formula is general, *i. e.*, that we have, for any positive integral value of n,

$$f(nx) = nf(x) + n(n-1)cx^2, \tag{F}$$

we assume that we have for $n = m$

$$f(mx) = mf(x) + m(m-1)cx^2 \tag{3}$$

and shall prove that (F) holds then for $n = m + 1$. On the strength of (f) we have

$$f[(m+1)x] = f(mx) + f(x) + 2cmx^2;$$

whence, substituting for $f(mx)$ its development from (3) and simplifying,

$$f[(m+1)x] = (m+1)f(x) + (m+1)mcx^2.$$

If in (f) we put $x = y = 0$, we have

$$f(0+0) = f(0) + f(0) - 2c\cdot 0\cdot 0$$

or $f(0) = 2f(0)$, hence $f(0) = 0$. Now for $y = -x$, (f) gives

$$f(x-x) = f(x) + f(-x) - 2cx^2$$

and since $f(0) = 0$, we have, solving for $f(-x)$,

$$f(-x) = -f(x) + 2cx^2. \tag{4}$$

This formula shows that (F) holds for $n = -1$. Hence in order to prove that (F) is valid for all negative integral values of n, it is sufficient to prove that it holds for $n = -(m+1)$, if it holds for $n = -m$. We assume therefore that

$$f(-mx) = -mf(x) + (-m)(-m-1)cx^2. \tag{5}$$

We have from (f)

$$f[-(m+1)x] = f(-mx) + f(-x) + 2c(-mx)(-x).$$

Replacing $f(-mx)$ and $f(-x)$ by their developments from (5) and (4), respectively, and simplifying, we obtain

$$f[-(m+1)x] = -(m+1)f(x) + [-(m+1)][-(m+1)-1]cx^2.$$

Since (F) is valid for all integral values of n, positive or negative, we have, k being any integer,

$$f(x) = f\left(k\cdot\frac{x}{k}\right) = kf\left(\frac{x}{k}\right) + k(k-1)c\left(\frac{x}{k}\right)^2, \tag{6}$$

which, when solved for $f(x/k)$, may be written in the form

$$f\left(\frac{1}{k}\cdot x\right) = \frac{1}{k}fx + \frac{1}{k}\left(\frac{1}{k}-1\right)cx^2. \tag{7}$$

Hence, (F) is valid for $n = 1/k$. Now let p, q be any two integers, positive or negative. We may write, since p is an integer,

$$f\left(p\cdot\frac{x}{q}\right) = pf\left(\frac{x}{q}\right) + p(p-1)c\frac{x^2}{q^2}$$

or, taking (7) into consideration and simplifying,

$$f\left(\frac{p}{q}\cdot x\right) = \frac{p}{q}fx + \frac{p}{q}\left(\frac{p}{q}-1\right)cx^2.$$

We have thus proved that (F) is valid for any rational value of n. Any irrational quantity s may be considered as the limit of a variable rational quantity, and since (F) holds for all these rational values, it also holds for s, if the function f is continuous, which we suppose. Hence, (F)

is valid for any real value of n. Since in the above considerations no restrictions were put upon x, (F) holds for any value of x, real or complex, if we assume, as we may, that (f) holds for such values of the variables. Thus we may write:

$$f(z) = f(u + vi) = f(u) + f(vi) + 2cuvi. \tag{8}$$

But

$$f(u) = f(u \cdot 1) = uf(1) + u(u - 1)c(1)^2,$$

$$f(vi) = f(v \cdot i) = vf(i) + v(v - 1)c(i)^2;$$

hence, (8) may be written:

$$f(z) = f(u + vi) = uf(1) + vf(i) + c(u + vi)^2 - (u - v)c. \tag{9}$$

In order to determine the value of $f(1)$ we put $u = 1$, and $v = 0$ and we have from (9) $f(1) = f(1)$, which shows that the value of $f(1)$ is arbitrary. By putting $u = 0$, $v = 1$,we may show in the same way that $f(i)$ is arbitrary. Hence denoting $f(1) - c$ by α and $f(i) + c$ by β, where α and β are two arbitrary quantities, (9) may be written

$$f(u + vi) = c(u + vi)^2 + \alpha u + \beta v.$$

If $\alpha = \beta = 0$, *i. e.*, if $f(1) = 1$ and $f(i) = -1$, and $c = 1$ we have

$$f(u + vi) = (u + vi)^2.$$

Also variously solved by T. M. Simpson, O. S. Adams, Elijah Swift, E. R. Smith, Horace Olson, A. A. Bennett, C. F. Gummer, and J. L. Walsh.

468. Proposed by H. C. Feemster, York College, Neb.

In each of the following series find the nth term and sum:

$$(a) \qquad 2 + 5 + 9 + 15 + 24 + \cdots,$$

$$(b) \qquad 1 + 6 + 10 + 20 + 35 + \cdots,$$

$$(c) \qquad 1 + 5 + 15 + 35 + 70 + \cdots,$$

Solution by J. L. Riley, Tahlequah, Okla.

(a) Using the method of differences we have

$$\begin{array}{cccccc} 3 & 4 & 6 & 9 & \cdots \\ & 1 & 2 & 3 & \cdots \\ & & 1 & 1 & \cdots \\ & & & 0 & \cdots \end{array}$$

$$U_n = 2 + 3(n-1) + \frac{(n-1)(n-2)}{\lfloor 2} + \frac{(n-1)(n-2)(n-3)}{\lfloor 3}$$

$$= \frac{n^3 - 3n^2 + 20n - 6}{6}, \text{ the } n\text{th term},$$

$$S_n = 2n + \frac{3n(n-1)}{\lfloor 2} + \frac{n(n-1)(n-2)}{\lfloor 3} + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}$$

$$= \frac{n}{24}(n^3 - 2n^2 + 35n + 14), \text{ the sum.}$$

(b) In the series $1 + 6 + 10 + 20 + 35 + \cdots$, let $U_n = A + Bn + Cn^2 + Dn^3 + En^4$. Then

$$\begin{cases} A + B + C + D + E = 1, \\ A + 2B + 4C + 8D + 16E = 6, \\ A + 3B + 9C + 27D + 81E = 10, \\ A + 4B + 16C + 64D + 256E = 20, \\ A + 5B + 25C + 125D + 625E = 35. \end{cases}$$

From these equations we find the values of A, B, C, D, E; whence,

$$U_n = -20 + 36n - \frac{115n^2}{6} + \frac{9n^3}{2} - \frac{n^4}{3}.$$

From this we get the 6th term to be 46.

Thus, the series is

$$1 + 6 + 10 + 20 + 35 + 46 + \cdots.$$

Using the method of differences we get

$$\begin{array}{ccccccc} 5 & 4 & 10 & 15 & 11 & \cdots \\ -1 & 6 & 5 & -4 & \cdot & \cdots \\ 7 & -1 & -9 & \cdot & \cdot & \cdots \\ -8 & -8 & \cdot & \cdot & \cdot & \cdots \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdots \end{array}.$$

Then

$$S_n = n + \frac{5n(n-1)}{\lfloor 2} - \frac{n(n-1)(n-2)}{\lfloor 3} + \frac{7n(n-1)(n-2)(n-3)}{\lfloor 4} - \frac{8n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5}$$

$$= -\frac{n}{120}(8n^4 - 115n^3 + 510n^2 - 1145n + 622).$$

(c) In the series $1 + 5 + 15 + 35 + 70 + \cdots$

$$U_n = A + Bn + Cn^2 + Dn^3 + En^4 + \cdots.$$

$$\left.\begin{array}{l} A + B + C + D + E = 1, \\ A + 2B + 4C + 8D + 16E = 5, \\ A + 3B + 9C + 27D + 81E = 15, \\ A + 4B + 16C + 64D + 256E = 35, \\ A + 5B + 25C + 125D + 625E = 70, \end{array}\right\} \begin{array}{l} A = 0, \\ B = 1/4, \\ C = 11/24, \\ D = 1/4, \\ E = 1/24. \end{array}$$

Hence,

$$U_n = \frac{n}{4} + \frac{11n^2}{24} + \frac{n^3}{4} + \frac{n^4}{24}.$$

Then

$$24S_n = 6\Sigma n + 11\Sigma n^2 + 6\Sigma n^3 + \Sigma n^4,$$

$$24S_n = 3n(n+1) + \frac{11n(n+1)(2n+1)}{6} + \frac{3n^2(n+1)^2}{2} + \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}.$$

$$S_n = \frac{n}{120}(n^4 + 10n^3 + 35n^2 + 50n + 24) = \frac{n}{120}(n+1)(n+2)(n+3)(n+4).$$

Also solved by O. S. ADAMS, H. H. CONWELL, PAUL CAPRON, and WILLIAM TIER.

GEOMETRY.

496. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

Find all the lines such that the pairs of tangent planes to a given sphere (ellipsoid) passing through them, shall be orthogonal.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

One obvious solution is given when one of the tangent planes is fixed in position, for then this plane is the locus of lines common to it and a second tangent plane orthogonal to the first.

It is not troublesome to show that the equation of a first plane embracing a line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}, \tag{1}$$

is

$$\frac{(x-a)\lambda_1}{l} + \frac{(y-b)\mu_1}{m} + \frac{(z-c)\nu_1}{n} = 0, \tag{2}$$

with the condition

$$\lambda_1 + \mu_1 + \nu_1 = 0, \tag{3}$$

and that of a second plane embracing (1) is

$$\frac{(x-a)\lambda_2}{l} + \frac{(y-b)\mu_2}{m} + \frac{(z-c)\nu_2}{n} = 0, \tag{4}$$

with the condition

$$\lambda_2 + \mu_2 + \nu_2 = 0. \tag{5}$$

If (2) and (4) are orthogonal,

$$\frac{\lambda_1}{l}\cdot\frac{\lambda_2}{l} + \frac{\mu_1}{m}\cdot\frac{\mu_2}{m} + \frac{\nu_1}{n}\cdot\frac{\nu_2}{n} = 0, \tag{6}$$

and if (2) and (4) touch

$$a_1x^2 + b_1y^2 + c_1z^2 = 1, \tag{7}$$

$$\left(\frac{\lambda_1}{l}\right)^2\Big/a_1 + \left(\frac{\mu_1}{m}\right)^2\Big/b_1 + \left(\frac{\nu_1}{n}\right)^2\Big/c_1 = \left(\frac{a\lambda_1}{l} + \frac{b\mu_1}{m} + \frac{c\nu_1}{n}\right)^2, \tag{8}$$

and

$$\left(\frac{\lambda_2}{l}\right)^2\Big/a_1 + \left(\frac{\mu_2}{m}\right)^2\Big/b_1 + \left(\frac{\nu_2}{n}\right)^2\Big/c_1 = \left(\frac{a\lambda_2}{l} + \frac{b\mu_2}{m} + \frac{c\nu_2}{n}\right)^2. \tag{9}$$

Now (3), (5), (6), (8), (9) is a system of *five* equations for the determination of the *six* ratios $\lambda_1/\nu_1, \mu_1/\nu_1$; $\lambda_2/\nu_2, \mu_2/\nu_2$; and $l/n, m/n$, giving an indeterminate solution. The values of $l/n, m/n$ are the only ones needed in (1).

There seems to be a missing condition in the statement of the problem.

If the direction of the line (1) were constant, or $l : m : n$, constant, the *locus* of the line (1) would be a right circular cylinder, or, in other words, the locus of the line of intersection of constant direction of pairs of orthogonal tangent planes to a central conicoid is a right circular cylinder.

Note.—The single condition on the line restricts it to be a member of a line complex whose order apparently may be as high as 8. It would be desirable to determine this explicitly. EDITORS.

497. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

Find the locus of the mid-point of the segment determined by two spheres on any line passing through any point common to the two spheres.

SOLUTION BY S. W. REAVES, University of Oklahoma.

Let the plane of the common circle of the spheres be chosen for yz-plane, and the line of centers for x-axis. Let the radius of the common circle be k, and let $(a, 0, 0)$ and $(b, 0, 0)$ be the centers of the two spheres. Then the equation of one sphere is

$$(x-a)^2 + y^2 + z^2 = a^2 + k^2,$$

or

$$x^2 + y^2 + z^2 - 2ax = k^2; \tag{1}$$

and, similarly, the equation of the other is

$$x^2 + y^2 + z^2 - 2bx = k^2. \tag{2}$$

Let l, m, n be the direction cosines of an arbitrary line through the point $M\ (0, 0, k)$. Then the equation of the line may be written

$$\frac{x}{l} = \frac{y}{m} = \frac{z-k}{n} = r, \tag{3}$$

where r is the length of the segment joining $(0, 0, k)$ and (x, y, z).

To find the length of the segment MP cut from this line by the sphere (1), we substitute $x = lr$, $y = mr$, $z = nr + k$ in equation (1) and solve for that value of r which is not zero. We thus find at once

$$r = MP = 2al - 2kn.$$

Substituting the same values in (2), we find likewise for the segment MQ intercepted by the

second sphere,

$$MQ = 2bl - 2kn.$$

If R be the mid-point of PQ, then

$$MR = \frac{MP + MQ}{2} = (a + b)l - 2kn.$$

Hence, if in (3) we set

$$r = (a + b)l - 2kn, \tag{4}$$

we readily obtain expressions for the coördinates x, y, z of the middle point R of the segment determined by the two spheres on the line through $(0, 0, k)$ and having the direction (l, m, n). To find the equation of the locus of R for all directions of the line we must eliminate l, m, n, which we do as follows:

From (3), we have,

$$x^2 + y^2 + (z - k)^2 = r^2; \tag{5}$$

and from (4) and (3),

$$r = (a + b)\left(\frac{x}{r}\right) - 2k\left(\frac{z - k}{r}\right),$$

or

$$r^2 = (a + b)x - 2k(z - k). \tag{6}$$

Equating the values of r^2 given by (5) and (6), we have for the equation of the locus of R,

$$x^2 + y^2 + z^2 - (a + b)x = k^2. \tag{7}$$

The locus is, therefore, a sphere whose center is midway between the centers of the given spheres and which contains all points common to these spheres. It is evident then that the choice of any point other than M common to the two spheres would lead to the same locus.

Also solved by the PROPOSER.

CALCULUS.

413. Proposed by OSCAR S. ADAMS, Washington, D. C.

Determine a function of x independent of b, such that

$$\int_b^{b+1} f(x)dx = \frac{1}{b + 1},$$

the real part of b being positive.

SOLUTION BY C. F. GUMMER, Kingston, Ontario.

There is one solution only satisfying the conditions:
(i) $f(x)$ is one-valued and continuous in the right half plane of complex numbers, and
(ii) $f(x + n)$ approaches 0 as n approaches $+\infty$ through integral values for every x.
Let $f(x)$ be such a solution. Differentiating the equation

$$\int_b^{b+1} f(x)dx = \frac{1}{b + 1}, \tag{1}$$

we get, after substituting x for b,

$$f(x + 1) - f(x) = -\frac{1}{(x + 1)^2}.$$

Similarly,

$$\begin{aligned} f(x + 2) - f(x + 1) &= -\frac{1}{(x + 2)^2}, \\ &\cdots\cdots\cdots \\ f(x + n) - f(x + n - 1) &= -\frac{1}{(x + n)^2}. \end{aligned}$$

Adding these equations, we have

$$f(x) = \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \cdots + \frac{1}{(x+n)^2} + f(x+n).$$

Letting n approach $+\infty$, we get, on account of (ii),

$$f(x) = \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \cdots \text{ to infinity.} \tag{2}$$

To show that (2) is a solution of (1), we observe that the series converges uniformly with respect to x over the right half plane including the axis of imaginaries, since $|x+n| \geq n$.

Hence,

$$\int_b^{b+1} f(x)dx = \left(\frac{1}{b+1} - \frac{1}{b+2}\right) + \left(\frac{1}{b+2} - \frac{1}{b+3}\right) + \cdots = \frac{1}{b+1},$$

for any path of integration lying in the above region.

The series (2) may be summed by means of the gamma function. Thus integrating between the limits from 0 to x, we have

$$\int_0^x f(x)dx = \left(\frac{1}{1} - \frac{1}{x+1}\right) + \left(\frac{1}{2} - \frac{1}{x+2}\right) + \cdots = S, \text{ say.}$$

Since S is uniformly convergent,

$$\begin{aligned}\int_0^x S dx &= \left\{\frac{x}{1} - \log\left(1 + \frac{x}{1}\right)\right\} + \left\{\frac{x}{2} - \log\left(1 + \frac{x}{2}\right)\right\} + \cdots \\ &= -\log\left[\left\{e^{-(x/1)}\left(1 + \frac{x}{1}\right)\right\}\left\{e^{-(x/2)}\left(1 + \frac{x}{2}\right)\right\}\cdots\right] \\ &= -\log\left(\frac{1}{\Gamma(x)xe^{\gamma x}}\right) = \log\Gamma(x+1) + \gamma x.\end{aligned}$$

Hence,

$$f(x) = \frac{dS}{dx} = \frac{d^2}{dx^2}[\log\Gamma(x+1)].$$

The general solution of (1) subject to (i) but not to (ii) is now seen to be

$$f(x) = \frac{d^2}{dx^2}\log\Gamma(x+1) + \frac{d}{dx}\phi(x),$$

where $\phi(b+1) = \phi(b)$, and, therefore, $\phi(x)$ is any analytic function admitting the period 1.

Also solved by W. R. RANSOM and the PROPOSER.

414. Proposed by C. N. SCHMALL, New York City.

Among spherical triangles having the same base and equal altitudes, show that the isosceles triangle has the greatest vertical angle. Show that this is also true for plane triangles.

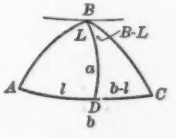
SOLUTION BY W. J. THOME, University of Detroit.

ABC is any spherical triangle. BD is a great circle arc perpendicular to AC. The angle B is broken up into two parts, L and $(B-L)$, and the corresponding opposite sides are l and $(b-l)$. Suppose B moves around on a small circle concentric with AC. Then BD remains constant in value.

In the $\triangle ABD$, $\cot L = \sin a \cot l$, and in the $\triangle BDC$, $\cot(B-L) = \sin a \cot(b-l)$. Now $B = L + (B-L) = \cot^{-1}(\sin a \cot l) + \cot^{-1}(\sin a \cot(b-l))$.

Differentiating,

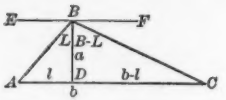
$$\frac{dB}{dl} = \sin a\left[\frac{-\csc^2 l}{1 + [\sin a \cot l]^2} + \frac{\csc^2(b-l)}{1 + [\sin a \cot(b-l)]^2}\right].$$



which is 0 if $b = 2l$. This evidently gives a maximum value of B. But if $b = 2l$, $AB = BC$ and the $\triangle$ is isosceles.

For the plane triangle, let ABC be any plane triangle. BD is a $\perp$ to AC from B. The vertex B may move anywhere on the straight line EF which is parallel to AC, thus keeping the altitude BD constant in value. The angle B is broken up into the two parts L and $(B - L)$ and the corresponding opposite sides are l and $(b - l)$.

In the $\triangle ABD$, $\tan L = l/a$ and in the $\triangle BDC$,



$$\tan (B - L) = \frac{b - l}{a}.$$

Now

$$B = L + (B - L) = \tan^{-1} (l/a) + \tan^{-1} \frac{b - l}{a}.$$

Differentiating,

$$\frac{dB}{dl} = a\left[\frac{1}{a^2 + l^2} - \frac{1}{a^2 + (b - l)^2}\right]$$

which is 0 if $b = 2l$, giving a maximum value of B. But if $b = 2l$, $AB = BC$ and the triangle is isosceles.

Also solved by M. R. GAFFET.

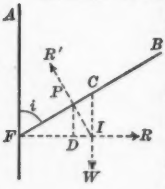
MECHANICS.

326. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A uniform beam, of length $2l$, rests in equilibrium against a smooth vertical wall and upon a peg at a distance a from the wall, show that the inclination of the beam to the vertical is

$$\sin^{-1}\left(\frac{a}{l}\right)^{\frac{1}{3}}.$$

SOLUTION BY H. S. UHLER, Yale University.



In order to obtain the given inclination it is necessary to assume that the reaction of the peg is normal to the beam, for, if this be not the case, the beam may be in equilibrium on various slants depending upon the value of the coefficient of friction.

The necessary and sufficient condition for static equilibrium is that the lines of action of the three forces acting on the beam pass through a single point, I. These forces are obviously the weight of the beam, W, acting vertically downward, the horizontal reaction of the *smooth* wall, R, and the normal reaction of the peg, R'. $\overline{BC} = \overline{CF} = l$. $\overline{DF} = a$. $P \equiv$ peg. Let $\overline{FP} \equiv m$ and $\overline{FI} \equiv n$. Since $\overline{IP}$ and $\overline{PD}$ are, respectively, the perpendiculars dropped from the vertices of the right angles upon the hypotenuses of the triangles CIF and FPI we have $n^2 = lm$ and $m^2 = an$, so that $a^2/m^2 = a/l$ or $a/m = (a/l)^{1/3}$. But $i = \sin^{-1} (a/m) = \sin^{-1} (a/l)^{1/3}$, as required.

Also solved by A. M. HARDING, L. A. WARREN, G. PAASWELL, PAUL CAPRON, O. S. ADAMS, and HORACE OLSON.

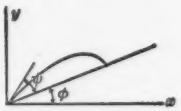
327. Proposed by C. N. SCHMALL, New York City.

An inclined plane makes an angle φ with the horizontal plane, and from its foot a body is projected upward at an angle ψ to the plane, and with a velocity v. Show that it will strike the plane perpendicularly if $\tan \psi = \frac{1}{2} \cot \varphi$ and that its range up the plane in this case will be

$$\frac{2v^2 \sin \varphi}{g(1 + 3 \sin^2 \varphi)}.$$

SOLUTION BY A. M. HARDING, University of Arkansas.

The position of the projectile at any time is given by



$$x = v\cos(\varphi+\psi)\cdot t,$$

$$y = v\sin(\varphi+\psi)\cdot t - \tfrac{1}{2}gt^2.$$

Eliminating t we obtain, as the equation of the path,

$$y = x\tan(\varphi+\psi) - \frac{gx^2}{2v^2\cos^2(\varphi+\psi)}. \tag{1}$$

Let r be the range up the plane. Then the coördinates of the point where the projectile meets the inclined plane are

$$x = r\cos\varphi, \qquad y = r\sin\varphi. \tag{2}$$

Substituting these in (1), we find

$$r = \frac{2v^2}{g}\cdot\frac{\cos(\varphi+\psi)\sin\psi}{\cos^2\varphi}. \tag{3}$$

From (1) we find

$$\frac{dy}{dx} = \tan(\varphi+\psi) - \frac{gt}{v\cos(\varphi+\psi)}. \tag{4}$$

This gives the direction at any time t. The projectile strikes the plane after a time

$$t = \frac{r\cos\varphi}{v\cos(\varphi+\psi)} = \frac{2v\sin\psi}{g\cos\varphi}.$$

Hence, when the projectile strikes the plane, we have

$$\frac{dy}{dx} = \tan(\varphi+\psi) - \frac{2\sin\psi}{\cos\varphi\cos(\varphi+\psi)}$$

$$= \frac{\sin(\varphi+\psi)\cos\varphi - 2\sin\psi}{\cos\varphi\cos(\varphi+\psi)}.$$

If the projectile strikes perpendicularly, $dy/dx = -\cot\varphi$. Hence,

$$\frac{\sin(\varphi+\psi)\cos\varphi - 2\sin\psi}{\cos\varphi\cos(\varphi+\psi)} = -\frac{\cos\varphi}{\sin\varphi}.$$

From this equation, we find $\tan\psi = \frac{1}{3}\cot\varphi$. Substituting in (3), we find

$$r = \frac{2v^2\sin\varphi}{g(1+3\sin^2\varphi)}.$$

Also solved by H. C. FEEMSTER, H. S. UHLER, G. PAASWELL, C. N. SCHMALL, O. S. ADAMS, HORACE OLSON, and L. A. WARREN.

NUMBER THEORY.

242. Proposed by NORMAN ANNING, Chilliwack, B. C.

Find a function of n which is equal to A_k when $n \equiv k \pmod p$, $k = 1, 2, 3, \cdots, p$.

SOLUTION BY THE PROPOSER.

Let θ be a primitive root of $x^p - 1 = 0$, then it is known that $\theta, \theta^2, \theta^3, \cdots, \theta^p$ are all the pth roots of unity and that $\theta^m + \theta^{2m} + \theta^{3m} + \cdots + \theta^{pm} = p$ or 0, according as m is or is not divisible by p. [See, for example, Burnside and Panton, *Theory of Equations*, Vol. I, pp. 95 and 96.] Consider now the expression:

$$f = \frac{1}{p}\,[A_1\{\theta^{n-1} + \theta^{2(n-1)} + \cdots + \theta^{p(n-1)}\}$$

$$+ A_2\{\theta^{n-2} + \theta^{2(n-2)} + \cdots + \theta^{p(n-2)}\}$$

$$+ \cdots\cdots\cdots\cdots$$

$$+ A_p\{\theta^{n-p} + \theta^{2(n-p)} + \cdots + \theta^{p(n-p)}\}].$$

Since, for any integral values of n, the numbers, $(n-1), (n-2), \cdots, (n-p)$, are p consecutive integers, one and only one of them is divisible by p.

Suppose $(n-k)$ is divisible by p, *i. e.*, $n \equiv k \pmod{p}$. Then the coefficient of A_k is p and all the other coefficients are zero.

Hence, when $n \equiv k \pmod{p}$, $f = A_k$ as required.

244. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Determine the rational value of x that will render $x^2 + px + q$ a perfect square. What value of x will render $x^2 - 7x + 2$ a perfect square?

SOLUTION BY HAROLD T. DAVIS, Colorado Springs, Colorado.

Let $x^2 + px + q = y^2$. Then $x^2 + px + (p^2/4) + q - (p^2/4) - y^2 = 0$, or $(2x+p)^2 - (2y)^2 = p^2 - 4q$.

(1) Let $2x + p = z$ and $2y = w$. Then $z^2 - w^2 = p^2 - 4q$; or, if $4q > p^2$, $w^2 - z^2 = 4q - p^2$. Let a and b be complementary factors of $p^2 - 4q$. Then $z + w = a$ and $z - w = b$. Whence $z = (a+b)/2$ and $w = (a-b)/2$. Substituting these values in equations (1), we have

$$x = \frac{a+b-2p}{4} \text{ and } y = \frac{a-b}{4}.$$

Example. $x^2 - 7x + 2 = y^2$. Here $p^2 - 4q = 41$, the complementary factors of which are 41 and 1. Hence,

$$x = \frac{41+1+14}{4} = 14 \quad \text{and} \quad y = \frac{41-1}{4} = 10.$$

A complete discussion of the solution of the general equation of the second degree in two variables is given in Chrystal's *Algebra*, Part II, page 458.

Also solved by HORACE OLSON, NORMAN ANNING, H. N. CARLETON, O. S. ADAMS, J. A. COLSON, J. L. RILEY, N. PANDYA and J. H. WEAVER.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

REPLIES.

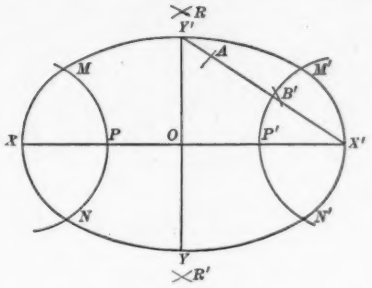
33. Under what conditions or to what extent is Mr. Iwerson's construction, given below, a useful or practical approximation to a true ellipse? What criterion can be given to measure definitely the degree of approximation?

Mr. Iwerson's approximate construction for an ellipse by ruler and compasses alone, having given the axes, was given in the November, 1916, issue of the MONTHLY, pp. 354, 355. The following corrections should be made: In the last two lines on p. 354, Ox should be OY, and Oy should be OX.

Note. In the February issue of the MONTHLY (pp. 90–92), we published a reply to this question by Professor Capron, of the U. S. Naval Academy. Before the February issue had come from the press, Professor Howland, of Wesleyan University, sent in the reply printed below. These two discussions are, accordingly, independent and from entirely different points of view. Professor Capron took as his primary measure of approximation the proportional errors in the radii of curvature at important points. Professor Howland has taken as his measure of approximation the ratio of the distance between the true and constructed curves (measured vertically or normally) to the semi-major axis. The two discussions seem to overlap in but one place. What Professor Capron has called the proportional error in the length of the minor axis and designated as E_1, corresponds to a maximum value of Professor Howland's relative divergence, d_2/a, which for some eccentricities occurs at $x = 0$. There is some slight difference in the formulas since in the one case the error is given relative to the semi-minor and in the other case relative to the semi-major axis.

The behavior of the arc MM', as brought out in the discussion below, is decidedly interesting. One would certainly not imagine *a priori* that the arc would cut the ellipse in four real distinct points.—U. G. M.

REPLY BY LEROY A. HOWLAND, Wesleyan University, Middletown, Conn.



In the figure, if the axes of the ellipse be taken as coördinate axes, the coordinates of the lettered points are readily found as follows:

$$X',\ (a, 0);\quad Y',\ (0, b);\quad B,\ \left(\frac{2ab^2}{a^2+b^2},\ \frac{a^2b-b^3}{a^2+b^2}\right);$$

$$A,\ \left(a-\frac{a^2}{\sqrt{a^2+b^2}},\ \frac{ab}{\sqrt{a^2+b^2}}\right);\quad M',\ \left(\frac{a+k}{2},\ \frac{(a-k)\sqrt{3}}{2}\right),$$

where

$$k=\frac{a^2-b^2}{\sqrt{a^2+b^2}};$$

$$P',\ (k, 0);\quad R',\ (0, -k\sqrt{3});\quad AB'=a-k;\quad MM'=a+k.$$

Since $\begin{vmatrix} \frac{a+k}{2}, & \frac{(a-k)\sqrt{3}}{2}, & 1 \\ k & 0 & 1 \\ 0 & -k\sqrt{3} & 1 \end{vmatrix} = 0$, M', P' and R' lie on a straight line

and since R' and P' are the centers of curvature of the arcs MM' and $M'N'$ respectively, the reason becomes evident why these arcs meet (in M') at an angle of 0°.

The equation of the ellipse, the circle about P', and the circle about R' are, respectively,

(1) $$x^2/a^2+y^2/b^2=1,$$

(2) $$(x-k)^2+y^2=(a-k)^2,$$

(3) $$x^2+(y+k\sqrt{3})^2=(a+k)^2.$$

We are, therefore, interested in the three arcs

$$y = \frac{b}{a}\sqrt{a^2 - x^2}, \tag{1'}$$

$$y = \sqrt{(a-k)^2 - (x-k)^2}, \qquad \frac{a+k}{2} \leqq x \leqq a, \tag{2'}$$

$$y = \sqrt{(a+k)^2 - x^2} - k\sqrt{3}, \qquad -\frac{a+k}{2} \leqq x \leqq \frac{a+k}{2}. \tag{3'}$$

The distances of the circular arcs from the ellipse are given by

$$d_1 = \sqrt{(a-k)^2 - (x-k)^2} - \frac{b}{a}\sqrt{a^2 - x^2},$$

$$d_2 = \sqrt{(a+k)^2 - x^2} - k\sqrt{3} - \frac{b}{a}\sqrt{a^2 - x^2}.$$

If we put $b/a = \lambda$, $x = ax'$, then

$$k = \frac{1-\lambda^2}{\sqrt{1+\lambda^2}}\,a,$$

and the ratios d_1/a and d_2/a, which we may call the relative divergences of the circular arcs, are functions of λ and x'.

If $a = b$ the construction is obviously exact, hence we shall assume $a > b$ or $\lambda < 1$.

Within its interval d_1 can easily be shown to be always greater than zero. To find its maximum, we have

$$\frac{d(d_1)}{dx} = -\frac{x-k}{\sqrt{(a-k)^2 - (x-k)^2}} + \frac{bx}{a\sqrt{a^2 - x^2}}.$$

The critical values are found to be $x = a$, which gives a minimum $d_1 = 0$, and the roots of the cubic

$$(a^2 - b^2)x^3 + (a^2 - b^2)(a - 2k)x^2 + a^2k(k - 2a)x + a^3k^2 = 0.$$

This cubic has three real roots, one in each of the intervals

$$(-\infty, 0), \quad \left(0, \frac{a+k}{2}\right), \quad \text{and} \quad \left(\frac{a+k}{2}, a\right).$$

There is, therefore, always one and only one critical value between $(a+k)/2$ and a. This gives a maximum value of d_1.

A discussion of d_2 can be made more complete, but is less simple, for the reason that in its interval it changes sign, in some cases as many as four times. To show this, we will find the intersections of

$$x^2/a^2 + y^2/b^2 = 1$$

and

$$x^2 + (y + k\sqrt{3})^2 = (a + k)^2.$$

Eliminating x^2 we have

$$(a^2 + b^2)y^2 - 2\sqrt{3}b^2\sqrt{a^2 + b^2}y + 2b^2(a\sqrt{a^2 + b^2} - a^2 + b^2) = 0,$$

whence

$$y = \frac{b^2\sqrt{3} \pm b(\sqrt{a^2 + b^2} - a)}{\sqrt{a^2 + b^2}}.$$

Both of these values of y can be shown to be greater than the ordinate of M' and hence all real intersections of these two curves occur along the arc under discussion. The corresponding values of x are given by the equations

$$x^2 = \frac{a^2(a - b\sqrt{3})[2\sqrt{a^2 + b^2} - a + b\sqrt{3}]}{a^2 + b^2},$$

$$x^2 = \frac{a^2(a + b\sqrt{3})[2\sqrt{a^2 + b^2} - a - b\sqrt{3}]}{a^2 + b^2}.$$

The expressions in brackets are always positive. Hence we have three distinct cases:

(1) $a < b\sqrt{3}$, two real distinct intersections;
(2) $a = b\sqrt{3}$, two real distinct and two coincident;
(3) $a > b\sqrt{3}$, four real distinct.

In case (1), d_2 becomes zero twice and changes sign twice. In case (2), d_2 becomes zero three times and changes sign twice. In case (3), d_2 becomes zero four times and changes sign four times. To find the maxima of d_2, we have

$$\frac{d(d_2)}{dx} = \frac{-x}{\sqrt{(a + k)^2 - x^2}} + \frac{bx}{a\sqrt{a^2 - x^2}}.$$

The critical values are

$$x_1 = -\sqrt{\frac{a^4 - b^2(a + k)^2}{a^2 - b^2}}, \qquad x_2 = 0, \qquad x_3 = \sqrt{\frac{a^4 - b^2(a + k)^2}{a^2 - b^2}},$$

$$\frac{d^2(d_2)}{dx^2} = -\frac{(a + k)^2}{[(a + k)^2 - x^2]^{3/2}} + \frac{ab}{(a^2 - x^2)^{3/2}},$$

$$\left.\frac{d^2(d_2)}{dx^2}\right|_{x=0} = -\frac{1}{a + k} + \frac{b}{a^2} = \frac{-a^2 + b(a + k)}{a^2(a + k)}.$$

This is evidently positive and gives a minimum so long as $a^2 < b(a + k)$, that is, so long as the critical points x_1 and x_3 are imaginary. Since d_2 is now negative, this minimum is a maximum of its absolute value. When $a^2 = b(a + k)$, the three critical points coincide at $x = 0$. It is found in this case that

$$\frac{d^2(d_2)}{dx^2} = \frac{d^3(d_2)}{dx^3} = 0,$$

but

$$\frac{d^4(d_2)}{dx^4} > 0,$$

and we still have a minimum d_2 or maximum $|\, d_2 \,|$. For $a^2 > b(a+k)$, $x = 0$ gives a maximum d_2, which has no interest for us until d_2 becomes positive for $x = 0$, *i. e.*, until $a > \sqrt{3}b$. This value is $a - b + (1 - \sqrt{3})k$ or, introducing $b/a = \lambda$,

$$\left|\frac{d_2}{a}\right|_{x=0} = 1 - \lambda + (1 - \sqrt{3})\frac{1 - \lambda^2}{\sqrt{1+\lambda^2}}.$$

This function of λ can be shown to have a negative derivative for $\lambda < 1/\sqrt{3}$, hence this maximum divergence increases as λ decreases and approaches as a limit

$$2 - \sqrt{3} = 0.268\cdots.$$

We also find that at x_1 and x_3, as soon as they become real, we have maxima for $|\, d_2 \,|$ equal to

$$k\sqrt{3} - \frac{\sqrt{a^2 - b^2}\,\sqrt{(a+k)^2 - a^2}}{a}.$$

or

$$\left|\frac{d_2}{a}\right|_{x=x_1} = \frac{1 - \lambda^2}{\sqrt{1+\lambda^2}}\left[\sqrt{3} - \sqrt{2\sqrt{1+\lambda^2} + 1 - \lambda^2}\right].$$

This function of λ is found to be an increasing function[1] for $0 < \lambda < 0.72\cdots$. Hence as λ decreases, $|\, d_2/a \,|$ decreases and approaches the limit 0.

The following typical cases indicate the degree of approximation of the construction:

λ	critical points	max. of $\left\|\frac{d_1}{a}\right\|$	max. of $\left\|\frac{d_2}{a}\right\|$
$\frac{3}{4}$	$x = 0.858a$	0.0222	——
	$x = 0$	——	0.0062
	$x = \pm\frac{\sqrt{-23}}{20}a$	——	——
$\frac{1}{\sqrt{3}}$	$x = 0.905a$	0.0217	——
	$x = 0$	——	——
	$x = \pm 0.506a$	——	0.0040
$\frac{5}{12}$	$x = 0.942a$	0.0156	——
	$x = 0$	——	0.0249
	$x = \pm 0.747a$	——	0.0015

[1] $a^2 > b(a+k)$, $1 > \lambda\left(1 + \frac{1-\lambda^2}{\sqrt{1+\lambda^2}}\right)$, $\sqrt{1+\lambda^2} > \lambda(1+\lambda)$, $\lambda^4 + 2\lambda^3 - 1 < 0$ or $\lambda < 0.72$.

The above discussion shows the vertical divergence between the true and the constructed ellipse. A better measure of the approximation, however, is the divergence along radii of the circular arcs.

If we transform our equation of the ellipse to axes through the point P' and then change to polar coördinates, we have:

$$\rho^2[a^2 - (a^2 - b^2)\cos^2\theta] + 2b^2k\rho\cos\theta + b^2(k^2 - a^2) = 0$$

or, introducing λ and letting $\cos\theta = t$,

$$\rho^2(1+\lambda^2)[1-(1-\lambda^2)t^2] + 2\lambda^2(1-\lambda^2)\sqrt{1+\lambda^2}\,at\rho + \lambda^2(\lambda^4 - 3\lambda^2)a^2 = 0.$$

This gives for the ellipse

$$\frac{\rho}{a} = -\frac{\lambda^2}{\sqrt{1+\lambda^2}}\left[\frac{(1-\lambda^2)t - \sqrt{3-\lambda^2-2(1-\lambda^2)t^2}}{1-(1-\lambda^2)t^2}\right].$$

For the circle about P' we have

$$\rho' = a - k = a\left(1 - \frac{1-\lambda^2}{\sqrt{1+\lambda^2}}\right)$$

or

$$\frac{\rho'}{a} = 1 - \frac{1-\lambda^2}{\sqrt{1+\lambda^2}}.$$

$\dfrac{\rho' - \rho}{a} = n_1$ we may call the relative normal divergence. The critical values of θ are found from the equation

$$\sqrt{1-t^2}\left\{[1-(1-\lambda^2)t^2]\left[1-\lambda^2-\frac{2(1-\lambda^2)t}{\sqrt{3-\lambda^2-2(1-\lambda^2)t^2}}\right]\right.$$
$$\left. + 2(1-\lambda^2)t[(1-\lambda^2)t - \sqrt{3-\lambda^2-2(1-\lambda^2)t^2}]\right\} = 0.$$

$t = 1$ or $\theta = 0$ gives us the minimum $n_1 = 0$. Equating the other factor to zero and simplifying, we obtain

$$2(1-\lambda^2)^2t^6 - (1-\lambda^2)(5-\lambda^2)t^4 + 2(2-\lambda^2)t^2 - 1 = 0$$

or

$$[(1-\lambda^2)t^2 - 1]^2[2t^2 - 1] = 0.$$

We have then the rather remarkable result that the value of θ giving a maximum n_1 is independent of λ and is always 45°. This maximum is

$$n_1 = 1 - \frac{1 + (\sqrt{2}-1)\lambda^2}{\sqrt{1+\lambda^2}},$$

and

$$\frac{d(\max n_1)}{d\lambda} = \frac{(\sqrt{2}-1)\lambda}{(1+\lambda^2)^{3/2}}[\sqrt{2} - 1 - \lambda^2].$$

As λ decreases, therefore, max n_1 increases until $\lambda^2 = \sqrt{2} - 1$, when it has the value $0.0148\cdots$. It then decreases, approaching 0 with λ.

In similar manner, the equation of the ellipse, referred to R' as pole and a horizontal through R' as initial line, is

$$\rho/a = \frac{1}{\sqrt{1+\lambda^2}}\left[\frac{\sqrt{3}(1-\lambda^2)s + \lambda\sqrt{2(1-\lambda^2)(2-\lambda^2)s^2 - (3-\lambda^2)(1-2\lambda^2)}}{(1-\lambda^2)s^2 + \lambda^2}\right],$$

where $s = \sin\theta$. For the circle $\rho'/a = 1 + \dfrac{1-\lambda^2}{\sqrt{1+\lambda^2}}$.

$|(\rho' - \rho)/a| = n_2$ will be the relative normal divergence for these arcs. Critical values of θ are found from an equation which in simplified form becomes

$$\sqrt{1-s^2}\,[(1-\lambda^2)s^2 + \lambda^2]^2\,[2(2-\lambda^2)s^2 - 3] = 0.$$

For $s = 1$, $\theta = 90°$, we have the vertical divergence discussed previously. The other real critical value is given by

$$s = +\sqrt{\frac{3}{2(2-\lambda^2)}}, \qquad \lambda^2 \leqq \tfrac{1}{2}.$$

The corresponding maximum is

$$|n_2| = \frac{\sqrt{2(2-\lambda^2)}}{\sqrt{1+\lambda^2}} - \frac{1-\lambda^2}{\sqrt{1+\lambda^2}} - 1.$$

The three typical cases, computed above, give the following results:

λ	max $\lvert n_1 \rvert$ $s^2 = \frac{1}{2}$	max $\lvert n_2 \rvert$ $s = 1$	max $\lvert n_2 \rvert$ $s = +\sqrt{\frac{3}{2(2-\lambda^2)}}$
$\frac{3}{4}$	0.0137	0.0062	——
$\frac{1}{\sqrt{3}}$	0.0145	0.0000	0.0038
$\frac{5}{12}$	0.0106	0.0249	0.0014

DISCUSSIONS.

Relating to the Indeterminate Form 0/0.

By M. O. Tripp, Olivet College, Olivet, Michigan.

In the May, 1916, number of the Monthly (Vol. XXIII, p. 180) Professor J. W. Nicholson considers the equation

$$y = \frac{x^2 - a^2}{x - a}, \tag{1}$$

and draws the conclusion that for $x = a$, *y has a definite and also an indeterminate value*. The object of this note is to show that we are not warranted in drawing such a conclusion.

When we clear (1) of fractions by multiplying both sides by $x - a$ and consider $(x - a)/(x - a)$ as having the value unity, we are at liberty to do so only upon condition that $x \neq a$.

Geometrically it is clear that when $x = a$ we are not warranted in drawing the conclusion that $y = 2a$ and $y = 0/0$. For, let x take a series of values from $x = a - d$ to $x = a + d$, $(d > 0)$. When $x \neq a$, $y = x + a$. Hence the coordinates of a point on the curve (1) satisfy the equation $y = x + a$, when $x \neq a$; but when $x = a$ we are not warranted in drawing the conclusion that y is necessarily equal to $2a$.

If we trace the locus of (1) from $x = a - d$ to $x = a + d$, we follow the straight line $y = x + a$ until we come to the point where $x = a$, then expand upward and downward to infinity vertically. As we pass on through $x = a$ the locus again follows the curve $y = x + a$. When $x = a$ in (1) we get the result that y is always genuinely indeterminate and does not take a definite value.

Note. The above discussion is published, as was Mr. Nicholson's, because of the interest which the subject possesses for many instructors.

In general, we take for granted that a function exists only where it has been defined. Consequently it seems to us that no amount of argument can get anywhere concerning the function in question for $x = a$ until the function has been defined for $x = a$, and when the function has been defined for $x = a$ the cause for argument disappears.

Neither Mr. Nicholson nor Mr. Tripp defines the function in question for $x = a$. In equation (1) of either paper, $y = x + a$ for $x \neq a$ and is not defined for $x = a$. Mr. Nicholson says (1) consists of two loci, but forgets that he introduces one of them, $x - a = 0$, when he multiplies both members of (1) by $x - a$.—U. G. M.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

After April 20, 1917, all communications to the Secretary, Professor W. D. CAIRNS, should be addressed to 55 East Lorain Street, Oberlin, Ohio, whither he is returning after a sojourn of seven months at the University of Chicago.

Dr. V. M. SLIPHER, for a number of years chief assistant at the Lowell Observatory, Flagstaff, Arizona, has been promoted to the directorship of the observatory, succeeding the late Percival Lowell.

Dr. EDWARD KIRCHER, Benjamin Peirce instructor in mathematics at Harvard University for the past two years, has accepted an instructorship in mathematics at the University of Minnesota.

CHARLES J. WHITE, emeritus professor of mathematics at Harvard University, died on February 12, at the age of seventy-eight years.

CHARLES A. PITKIN, professor of mathematics and physics at Thayer Academy, South Braintree, Mass., since 1877, died on November 7, at the age of sixty-three years.

Professor FLORIAN CAJORI, president of the Mathematical Association of America, has a review of Macfarlane's "Lectures on Ten British Mathematicians of the Nineteenth Century" in *Science*, January 26.

According to press dispatches from Berlin, the Prussian Minister of Education has announced to the Budget Committee of the Reichstag that 10,950 public-school teachers have fallen during the present European war and that their places have been taken by women.

Professor L. E. DICKSON, of the University of Chicago, has retired from the editorial committee of the *Transactions of the American Mathematical Society*, his unexpired term being filled by the selection of Professor L. P. EISENHART. Professors G. A. BLISS and E. B. WILSON, associate editors of the *Transactions*, have also retired, and have been succeeded by Professors C. N. MOORE and F. R. SHARPE.

The prize awards by the Paris Academy of Science, 1916, included 2,000 francs to Professor N. E. NÖRLUND, of the University of Lund, for his work on linear difference equations. The Paris Academy also announces the following mathematical prizes for 1918: (1) The Poncelet prize of 2,000 francs, to the author of the work most useful to the progress of pure mathematics; (2) The Francoeur prize of 1,000 francs, for discoveries or works useful to the progress of pure or applied mathematics.

The Texas Mathematics Teachers' Bulletin, Vol. 2, No. 2, appeared on December 15, 1916. This periodical is edited by Adjunct Professor J. W. CALHOUN, and Associate Professor C. D. RICE, of the University of Texas. It is published as a *Bulletin* of the University of Texas, and is open to the teachers of mathematics in Texas for the expression of their views, the editors assuming no responsibility for statements of facts or opinions in articles not written by them. The contents of the present number will show the nature of the subjects discussed. The following subjects are treated: "The geometry original," by J. G. DUNLAP; "The mathematics of investment," by E. L. DODD; "Literal arithmetic," by C. D. RICE; "On postulational systems," A. A. BENNETT.

The British Mathematical Association held its annual meeting at London on January 5, at which the following papers were presented: "The school syllabus in geometry," by T. P. NUNN; "Some of the work of the teaching committee," by A. W. SIDDONS; "Technical education and its relations to literature and science," by A. N. WHITEHEAD; "An accuracy test set in some public schools," by A. W. SIDDONS; "The place of mathematics in education reconstruction," by P. ABBOTT.

Regular meetings of the Edinburgh Mathematical Society were held on December 8 and January 12 at which the following papers were read: "On a class of continued fractions" and "A method of solving algebraic equations," by L. R. FORD; "On certain determinants of Cayley and Sylvester," by E. T. WHITTAKER; "An addition to the slide rule," by E. M. HORSBURGH; "The apolar locus of two tetrads of points," by W. P. MILNE.

At the January meeting of the London Mathematical Society the following papers were presented: "Asymptotic formulæ in combinatory analysis," by G. H. HARDY and S. RAMANUJAN; "The singular solutions of ordinary differential equations of the first order," by M. J. M. HILL; "The nature of a moving electric charge and its lines of electric force," by H. BATEMAN; "The expansion of the variables of a hypergeometric equation in terms of the ratio of two solutions," by L. J. ROGERS; "A problem in the theory of diffraction," by H. J. PRIESTLEY.

The February number of the *Proceedings* of the National Academy of Sciences contains the following mathematical papers: "Natural and isogonal families of curves on a surface," by JOSEPH LIPKA; "Some problems of Diophantine approximation," by G. H. HARDY and J. E. LITTLEWOOD; "A note on the fitting of parabolas," by J. R. MINER.

With this issue of the MONTHLY is completed a synopsis of mathematical courses offered in the coming summer sessions of the various colleges and universities of this country. Eighteen programs have been received in response to the request in the February number of the MONTHLY and the circular letter issued in January. The following summer announcements have been received since the copy for the March number of the MONTHLY was made up:

UNIVERSITY OF CALIFORNIA. Summer session, June 25–August 4. By Professor T. M. PUTNAM: Functions of a complex variable; Seminar for graduate students.—By Professor E. R. HEDRICK (University of Missouri): Functions of a real variable; A survey of elementary mathematics.—By Professor B. M. WOODS: Mathematics of investment; Integral calculus.—By Dr. H. N. WRIGHT: Synthetic projective geometry; Differential geometry. The elementary courses in freshman mathematics will be given.

HARVARD UNIVERSITY. Summer session, July 2–August 11. By Professor M. BÔCHER: Plane analytic geometry, five hours.—By Professor D. JACKSON: Logarithms and trigonometry, five hours.

UNIVERSITY OF KENTUCKY. Summer session, June 11–July 25. By Professor P. P. BOYD: Integral calculus, six hours; Analytics, six hours; College algebra, six hours.—By Professor J. M. DAVIS: Differential calculus, six hours: Trigonometry, six hours. Courses are offered in high-school algebra and geometry.

THE UNIVERSITY OF MINNESOTA. Summer session, June 18–July 28. By Professor G. N. BAUER: Trigonometry, eight hours; Fundamental concepts of secondary mathematics, four hours.—By Professor W. H. KIRCHNER: Solid geometry, eight hours; Descriptive geometry, eight hours.—By Professor W. F. HOLMAN: Integral calculus, twelve hours.—By Professor W. D. REEVE: Teachers' course, eight hours; Algebra, eight hours.—By Professor H. L. SLOBIN: Theory of equations, eight hours; Differential calculus, eight hours.—By Mr. R. M. BARTON: Analytics, eight hours; Algebra, eight hours.

UNIVERSITY OF MISSOURI. Summer session, June 7–August 3. By Professor W. D. A. WESTFALL: Integral calculus, five hours; Calculus of variations, three hours; Synoptic course, three hours.—By Professor W. D. CAIRNS (Oberlin College): Analytic geometry, five hours; Elementary calculus, five hours; Advanced calculus, three hours.—By Professor L. INGOLD: College algebra, five hours; Trigonometry, five hours; Teaching of mathematics, three hours.

UNIVERSITY OF SOUTHERN CALIFORNIA. Summer session, July 2–August 11. By Professor PAUL ARNOLD: Teachers' course, five hours; Integral calculus, five hours.—By Professor H. C. WILLETT: Analytic geometry, five hours; Trigonometry, five hours.

THE UNIVERSITY OF TENNESSEE. Summer session, June 19–July 27. By Professor H. E. BUCHANAN: History of mathematics; Trigonometry; Teachers' course in algebra.—By Professor J. B. HAMILTON: College algebra; Solid geometry; Plane geometry; Elementary algebra.

UNIVERSITY OF WASHINGTON. Summer session, June 18–July 31. By Professor R. E. MORITZ: Elementary graphs, five hours; Mathematics of investment, five hours; Definite integrals, five hours.—By Professor A. F. CARPENTER: Trigonometry, five hours; Mathematics in the high schools, five hours.—By Professor E. T. BELL: Teachers' course in algebra, five hours; Theory of equations, five hours.

KIRKSVILLE (MO.) STATE NORMAL SCHOOL. Summer Session, May 30–Aug. 2. By Professor WM. H. ZEIGEL: Integral Calculus and Surveying.—By Professor BYRON COSBY: Teaching of Elementary Mathematics; Trigonometry.—By Professor G. H. JAMISON: College Algebra; Analytic Geometry.—By Professor CHAS. A. EPPERSON: Teaching of Secondary Mathematics; Theory of Equations.

Various members of the Association are announced as special instructors for the summer sessions in institutions other than their own. Among these are Professors W. D. CAIRNS, at the University of Missouri, E. R. HEDRICK, at the University of California, D. N. LEHMER and G. D. BIRKHOFF, at the University of Chicago, A. COHEN and B. F. FINKEL, at the University of Colorado.

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THE ORIGIN OF MATHEMATICAL INDUCTION.*

By W. H. BUSSEY, University of Minnesota.

INTRODUCTION.

A criticism often made of mathematics as a subject of study in our high schools and colleges is that it involves nothing of observation, experiment, and induction as these terms are understood in the natural sciences. Whether or not the old and well-developed branches of mathematics as taught in our schools have been made into such well-organized deductive disciplines that this criticism is just, it is true that the work of the original investigators who have developed mathematical science has involved a great deal of observation, experiment, and induction; induction being, according to the Century Dictionary, "the process of drawing a general conclusion from particular cases." Observation and experiment in mathematics do not involve costly and complicated apparatus as often in physics, astronomy, and the other sciences, pencil and paper being all that is ordinarily necessary, but they are just as truly observation and experiment.

In the natural sciences a law arrived at by observation and experiment has to be verified by subsequent experiment by the same or other observers, either directly by repetition of the same experiment or indirectly by testing some logical consequence of the law in question. But in mathematics it is often possible to give rigorous demonstrations of theorems arrived at by ordinary induction. One method of clinching an argument by ordinary induction is what has been called *mathematical induction*. A more significant name and one that is being used more and more is *complete induction*. It is not a method of discovery but a method of proving rigorously that which has already been discovered. It is one of the most fruitful methods in all mathematics. It has

* Read before the Minnesota Section of the Association at its second meeting, April 9, 1917.